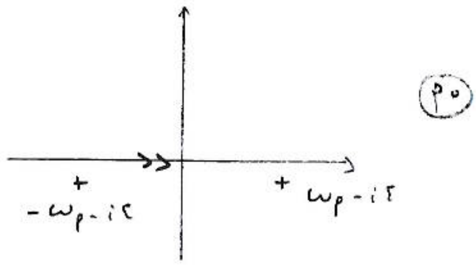
# Propagation et radiation en théorie classique des champs

## 1. Fonction de Green

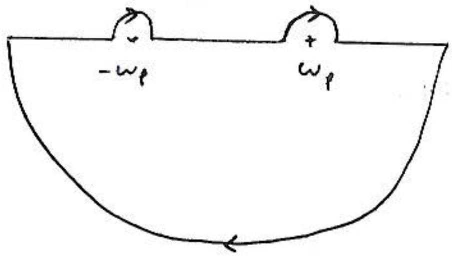
1°) $G_{\substack{Ret\\AV}}(x) = \dfrac{-1}{(2\pi)^4}\displaystyle\int d^4p\; e^{-ip.x}\, \dfrac{1}{(p_0 \pm i\varepsilon)^2 - \vec{p}^2 - m^2}$

$$\frac{1}{(p_0 + i\varepsilon)^2 - \vec{p}^2 - m^2} = \frac{1}{2\omega_p}\left[\frac{1}{p_0 - \omega_p + i\varepsilon} - \frac{1}{p_0 + \omega_p + i\varepsilon}\right]$$

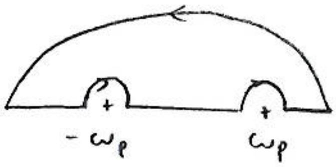
avec $\omega_p = \sqrt{\vec{p}^2 + m^2}$




$x_0 - y_0 > 0$ :




$x_0 - y_0 < 0$ :




$\longrightarrow 0$

donc $G_{Ret}(x-y) = \theta(x_0 - y_0)\,\dfrac{2i\pi}{(2\pi)^4}\displaystyle\int d^3\vec{p}\left[\frac{e^{-i\omega_p(x_0-y_0) + i\vec{p}(\vec{x}-\vec{y})}}{2\omega_p}\right.$

$$\left. - \frac{e^{i\omega_p(x_0-y_0) - i\vec{p}(\vec{x}-\vec{y})}}{2\omega_p}\Bigg|_{\vec{p}\to-\vec{p}}\right]$$

$$G_{Ret}(x-y) = \frac{i\,\theta(x_0-y_0)}{(2\pi)^3}\int \frac{d^3p}{2\omega_p}\left[e^{-ip(x-y)} - e^{ip(x-y)}\right]_{p_0 = \omega_p}$$

rem: $D(x-y) = \dfrac{1}{(2\pi)^3}\displaystyle\int \frac{d^3p}{2\omega_p}\, e^{-ip(x-y)} = \langle 0|\,\phi(x)\,\phi(y)\,|0\rangle$

en théorie quantique des champs bosoniques

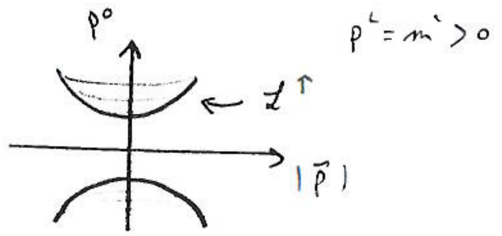
$$G_{Ret}(x-y) = i\,\Theta(x_0-y_0)\big(D(x-y) - D(y-x)\big)$$
$$= i\,\Theta(x_0-y_0)\,[\phi(x),\phi(y)]$$
$$= i\,\Theta(x_0-y_0)\,\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$D(x,y)$ est invariant par $\mathcal{L}^{\uparrow}$ (Lorentz orthochrone) :

$$D(\Lambda x,\Lambda y) = \int \frac{d^3p}{2\omega_p}\,\frac{1}{(2\pi)^3}\,e^{-ip\cdot\Lambda(x-y)}$$

$$p\cdot\Lambda v = \Lambda^{-1}p\cdot v$$

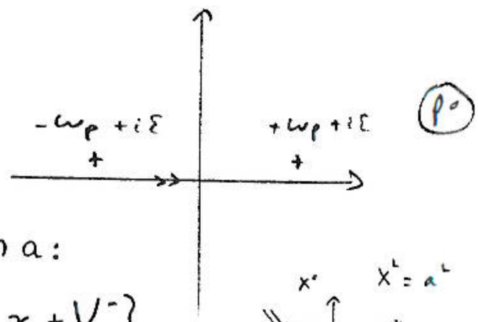
$$\int\frac{d^3p}{2\omega_p} = \int d^4p\;\delta(p^2-m^2)\,\Theta(p^0) \quad \text{est invariant sous } \mathcal{L}^{\uparrow}$$




Donc $D(\Lambda x,\Lambda y) = D(x,y)$ pour $\Lambda\in\mathcal{L}^{\uparrow}$

d'où $G_{ret}(\Lambda(x_0-y_0)) = G_{ret}(x_0-y_0)$
pour $\Lambda\in\mathcal{L}^{\uparrow}$

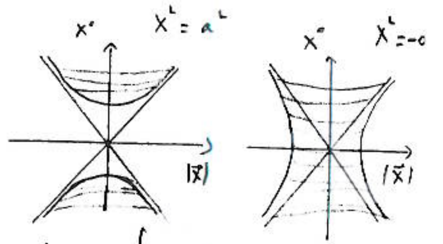
De même pour $G_{av}$ :




$$G_{av}(x-y) = i\,\Theta(y_0-x_0)\big(D(y-x) - D(x-y)\big)$$

Comme $D(x-y)$ est invariante par $\mathcal{L}^{\uparrow}$, on a :

* $G_{ret}(x-y)$ : support sur $\{y\in\mathbb{R}^4 / y\in x+V^-\}$
* $G_{av}(x-y)$ : " " $\{y\in\mathbb{R}^4 / y\in x+V^+\}$




$V^+ = \{x\in\mathbb{R}^4 / x^0\geq 0,\ x^2\geq 0\}$ demi-cône de lumière futur

$V^- = -V^+$ demi-cône de lumière passé

2°) $m^2=0$ $\quad G_{ret}(x) = i\,\Theta(x_0)[D(x) - D(-x)]$

avec $D(x) = \frac{1}{(2\pi)^3}\int\frac{d^3\vec p}{2p}\,e^{-ipx_0 + i\vec p\cdot\vec x - x_0\varepsilon}$ vient du pôle en $\omega_p - i\varepsilon$

$D(-x) = D^*(x) = \dots \int d^3\vec p\; e^{ipx_0 - i\vec p\cdot\vec x - x_0\varepsilon}$ [illegible]

$x_0>0$ $$\mathcal{D}(x) = \frac{1}{(2\pi)^3}\int \frac{1}{2p} e^{-ipx_0 + ipx\cos\theta - x_0\varepsilon} 2\pi p^2 dp\, d\theta \sin\theta \qquad x = \|\vec{x}\|$$

$$= -\frac{2\pi}{2(2\pi)^3}\int_0^{+\infty}\int_0^{\pi} e^{-ip(x_0 - i\varepsilon) + ipx\cos\theta} p\,dp\, d(\cos\theta)$$

$$= +\frac{1}{8\pi^2}\int_0^{+\infty} p\,dp\int_{-1}^{1} e^{-ip(x_0 - i\varepsilon) + ipxy} dy = \frac{1}{ix8\pi^2}\int_0^{+\infty}\left(e^{-ip(x_0 - x - i\varepsilon)} - e^{-ip(x_0 + x - i\varepsilon)}\right)$$

$$= \frac{1}{i|x|8\pi^2}\left[\frac{-1}{i(|\vec{x}| + x_0 - i\varepsilon)} + \frac{1}{i(x_0 - |\vec{x}| - i\varepsilon)}\right] = \frac{1}{8\pi^2|x|}\left(\frac{1}{|\vec{x}| + x_0 - i\varepsilon} + \frac{1}{|\vec{x}| - x_0 + i\varepsilon}\right)$$

Comme $\dfrac{1}{x \pm i\varepsilon} = PP\dfrac{1}{x} \mp i\pi\delta(x)$, on en déduit:

$$G_{ret}(x) = i\,\Theta(x_0)\left(\mathcal{D}(x) - \mathcal{D}^*(x)\right)$$

$$= \frac{i\,\Theta(x_0)}{8\pi^2|x|}\left[\frac{1}{|\vec{x}| + x_0 - i\varepsilon} - \frac{1}{|\vec{x}| + x_0 + i\varepsilon} + \frac{1}{|\vec{x}| - x_0 + i\varepsilon} - \frac{1}{|\vec{x}| - x_0 - i\varepsilon}\right]$$

$$= \Theta(x_0)\frac{1}{4\pi|\vec{x}|}\left[-\delta(|\vec{x}| + x_0) + \delta(|\vec{x}| - x_0)\right]$$

$$= \frac{\Theta(x_0)}{4\pi|\vec{x}|}\delta(|\vec{x}| - x_0)$$

$$G_{ret}(x) = \frac{1}{2\pi}\Theta(x_0)\,\delta(x^2)$$

de même $$G_{av}(x) = i\,\Theta(-x_0)\left(\mathcal{D}(-x) - \mathcal{D}(x)\right)$$

$$= \frac{\Theta(-x_0)}{4\pi|\vec{x}|}\left[\delta(|\vec{x}| + x_0) - \delta(|\vec{x}| - x_0)\right]$$

$$= \frac{\Theta(-x_0)}{4\pi|\vec{x}|}\delta(|\vec{x}| + x_0)$$

$$G_{av}(x) = \frac{1}{2\pi}\Theta(-x_0)\,\delta(x^2)$$

$$\boxed{G_{\substack{ret\\av}}(x)\Big|_{m^2=0} = \frac{1}{2\pi}\Theta(\pm x_0)\,\delta(x^2)}$$

support sur le bord du cône de lumière futur ($G_{ret}$) passé ($G_{av}$).

3°) avec des conditions aux limites périodiques, la condition de quantification s'écrit :

$$\varphi_{\pm,\vec{p}}(x^\circ, L\vec{u}_1 + \vec{x}) = \varphi_{\pm,\vec{p}}(x^\circ, L\vec{u}_2 + \vec{x})$$

$$= \varphi_{\pm,\vec{p}}(x^\circ, L\vec{u}_3 + \vec{x}) = \varphi_{\pm,\vec{p}}(x^\circ, \vec{x})$$

$$\text{soit } e^{ip_1L} = e^{ip_2L} = e^{ip_3L} = 0$$

$$\Rightarrow \vec{p} = (p_1, p_2, p_3) = \frac{2\pi}{L}\vec{n} \quad n \in \mathbb{Z}^3$$

donc $\displaystyle \frac{1}{L^3} \sum_{\vec{p}=\frac{2\pi\vec{n}}{L},\, \vec{n}\in\mathbb{Z}^3} \underset{L\to+\infty}{\longrightarrow} \frac{1}{(2\pi)^3}\int d^3\vec{p}$

comme $\displaystyle \varphi_{\pm,\vec{p}}(x) = \frac{1}{L^{3/2}\sqrt{2\omega_p}} e^{\mp i\omega_p x_0 + i\vec{p}\cdot\vec{x}}$, on obtient

$$G_{ret}(x-y) = \lim_{L\to\infty} i\,\Theta(x^\circ - y^\circ)\sum\Big[\underbrace{\varphi_{+,\vec{p}}(x)\,\varphi^*_{+,\vec{p}}(y)}_{①} - \underbrace{\varphi_{-,\vec{p}}(x)\,\varphi^*_{-,\vec{p}}(y)}_{② \,=\, \left(\varphi_{+,\vec{p}}(x)\,\varphi^*_{+,\vec{p}}(y)\right)^*}\Big]$$

$\rightarrow G_{ret}(x-y)$ est réelle

(immédiat en fait car

$G_{ret}(x-y) = i\,\Theta(x^\circ - y^\circ)\left[D(x-y) - D^*(x-y)\right]$)

$G_{ret}$ propage donc les fréquences positives (①) et les fréquences négatives (②) vers le futur.

4°) $G^{(-)}(x-y) = G_{ret}(x-y) - G_{av}(x-y)$

$$= i\frac{\Theta(x_0-y_0)}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,\Theta(p^0)\left(e^{-ip(x-y)} - e^{ip(x-y)}\right)$$

$$-i\frac{\Theta(y_0-x_0)}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,\Theta(p^0)\left(e^{ip(x-y)} - e^{-ip(x-y)}\right)$$

$$= \frac{i}{(2\pi)^3}\underbrace{\left[\Theta(x_0-y_0) + \Theta(y_0-x_0)\right]}_{1}\int d^4p\,\delta(p^2-m^2)\,\Theta(p^0)\,e^{-ip(x-y)}$$

$$- \frac{i}{(2\pi)^3}\left[\Theta(x_0-y_0) + \Theta(y_0-x_0)\right]\int d^4p\,\delta(p^2-m^2)\,\Theta(p^0)\,e^{ip(x-y)}$$

Le deuxième terme s'écrit encore, en faisant $p \to -p$ :

$$\frac{-i}{(2\pi)^3}\int d^4p\, \delta(p^2-m^2)\,\Theta(-p^0)\, e^{-ip(x-y)}$$

d'où $$G^{(-)}(x) = \frac{i}{(2\pi)^3}\int d^4p\, e^{-ip.x}\,\varepsilon(p_0)\,\delta(p^2-m^2)$$ où $\varepsilon(p_0) = \mathrm{sgn}(p_0)$

donc $G^{(-)}(x)$ est <u>impaire</u> ($x \to -x$ : correspond à faire $p \to -p$ dans l'intégrale donc $\varepsilon(p_0) \to -\varepsilon(p_0)$)

$G^{(-)}$ s'écrit aussi :

$$G^{(-)}(x) = \frac{i}{(2\pi)^3}\int \frac{d^3p}{2\omega_p}\left(e^{-i\omega_p x_0 + i\vec{p}\cdot\vec{x}} - e^{i\omega_p x_0 + i\vec{p}\cdot\vec{x}}\right)$$

Comme $$\begin{cases}(\Box+m^2)\,G_{av}(x) = \delta^4(x)\\ (\Box+m^2)\,G_{ret}(x) = \delta^4(x)\end{cases}$$

il est immédiat que $(\Box+m^2)\,G^{(-)}(x) = 0$

<u>support</u> : $G_{ret}(x)$ : $x \in V^+$

$G_{av}(x)$ : $x \in V^-$

donc $G^{(-)}(x)$ a son support sur $V = V^- \cup V^+$ = cône de lumière

de $$G^{(-)}(x) = \frac{i}{(2\pi)^3}\int\frac{d^3p}{2\omega_p}\left(e^{-i\omega_p x_0 + i\vec{p}\cdot\vec{x}} - e^{i\omega_p x_0 + i\vec{p}\cdot\vec{x}}\right)$$

on déduit : $$\left.\frac{\partial}{\partial x_0}G^{(-)}(x)\right|_{x^0=0} = \frac{i}{(2\pi)^3}\int\frac{d^3p}{2\omega_p}\left(-i\omega_p e^{i\vec{p}\cdot\vec{x}} - i\omega_p e^{i\vec{p}\cdot\vec{x}}\right)$$

$$= \frac{1}{(2\pi)^3}\int d^3p\; e^{i\vec{p}\cdot\vec{x}} = \delta^3(\vec{x})$$

si $m^2=0$ : $$G^{(-)}(x) = G_{ret}(x) - G_{av}(x)$$

$$= \frac{1}{2\pi}\left[\Theta(x_0) - \Theta(-x_0)\right]\delta(x^2) = \frac{1}{2\pi}\,\varepsilon(x_0)\,\delta(x^2)$$

$$\frac{1}{(x-i\eta)^2} - \frac{1}{(x+i\eta)^2} \sim \frac{1}{x^2 - 2ix_0\eta_0} - \frac{1}{x^2+2ix_0\eta_0}$$

$\eta = (\eta_0, 0, 0, 0)$

<u>$x_0>0$</u> : $$\frac{1}{x^2-i\varepsilon} - \frac{1}{x^2+i\varepsilon} = +2i\pi\,\delta(x^2)$$

<u>$x_0<0$</u> : $$\frac{1}{x^2+i\varepsilon} - \frac{1}{x^2-i\varepsilon} = -2i\pi\,\delta(x^2)$$

Donc $G^{(-)}(x)\big|_{m^2=0} = \lim_{\eta\to 0} \frac{1}{4\pi^2 i}\left[\frac{1}{(x-i\eta)^2} - \frac{1}{(x+i\eta)^2}\right]$

$$G^{(+)}(x) = \frac{1}{2}\left[G_{ret}(x) + G_{av}(x)\right]$$

$$= -\frac{1}{2}\frac{1}{(2\pi)^4}\int d^4p\, e^{-ipx}\left[\frac{1}{(p_0+i\varepsilon)^2-\vec{p}^{\,2}-m^2}+\frac{1}{(p_0-i\varepsilon)^2-\vec{p}^{\,2}-m^2}\right]$$

$$= -\frac{1}{2}\frac{1}{(2\pi)^4}\int d^4p\, e^{-ipx}\frac{1}{2\omega_p}\left(\frac{1}{p_0-\omega_p+i\varepsilon}-\frac{1}{p_0+\omega_p+i\varepsilon} + \frac{1}{p_0-\omega_p-i\varepsilon}-\frac{1}{p_0+\omega_p-i\varepsilon}\right)$$

$$= -\frac{1}{2}\frac{1}{(2\pi)^4}\int d^4p\, e^{-ipx}\frac{1}{2\omega_p}\, PP\left(\frac{2}{p_0-\omega_p}-\frac{2}{p_0+\omega_p}\right)$$

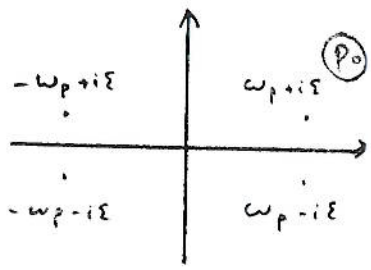
$$G^{(+)}(x) = -\frac{PP}{(2\pi)^4}\int d^4p\, e^{-ipx}\frac{1}{p^2-m^2}$$ vérifie $G^{(+)}(x)=G^{(+)}(-x)$

5°) propagateur de Feynman :

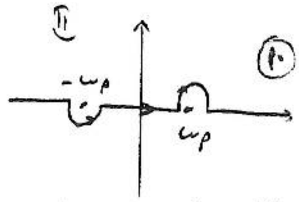
$$G_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ipx}\frac{1}{p^2-m^2+i\varepsilon}$$

$$\frac{1}{p^2-m^2+i\varepsilon} = \frac{1}{(p_0+i\varepsilon)^2-\vec{p}^{\,2}-m^2}\Theta(\mathrm{Re}\,p_0) + \frac{1}{(p_0-i\varepsilon)^2-\vec{p}^{\,2}-m^2}\Theta(-\mathrm{Re}\,p_0)$$

$$= \frac{\Theta(\mathrm{Re}\,p_0)}{2\omega_p}\left[\frac{1}{p_0-\omega_p+i\varepsilon} - \cancel{\frac{1}{p_0+\omega_p+i\varepsilon}}\right] + \frac{\Theta(-\mathrm{Re}\,p_0)}{2\omega_p}\left[\cancel{\frac{1}{p_0-\omega_p-i\varepsilon}} - \frac{1}{p_0+\omega_p-i\varepsilon}\right]$$




donc $$G_F(x-x') = \frac{i}{(2\pi)^3}\int\frac{d^3p}{2\omega_p}\left[\Theta(x_0-x_0')\,e^{-i\omega_p(x_0-x_0')+i\vec{p}(\vec{x}-\vec{x}')} + \Theta(x_0'-x_0)\,e^{i\omega_p(x_0-x_0')+i\vec{p}(\vec{x}-\vec{x}')}\right] = i\,\Theta(x_0-x_0')\,D(x_0-x_0') + i\,\Theta(x_0'-x_0)\,D(x_0'-x_0)$$




$$:(x-x') = \lim_{L\to+\infty} i\left\{\Theta(x_0-x_0')\sum \varphi_{+,\vec{p}}(x)\,\varphi^*_{+,\vec{p}}(x') + \Theta(x_0'-x_0)\sum\varphi_{-,\vec{p}}(x)\,\varphi^*_{-,\vec{p}}(x')\right\}$$

Les solutions d'énergie positive se propagent vers le futur

" " " négative " " " le passé

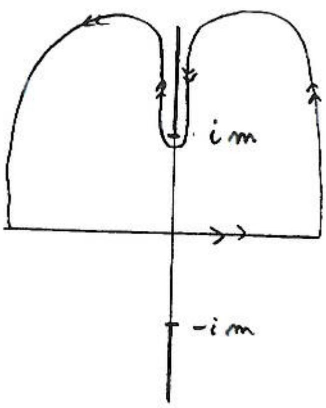
→ distinction particule / antiparticule pour des particules chargées (cette charge peut être la charge électrique : $\pi^+/\pi^-$, la charge d'étrangeté : $K^0/\bar{K}^0$, etc...)

$G_{ret}$, $G_{av}$, $G^{(-)}$, $G^{(+)}$ s'annullent en dehors du cône de lumière

en revanche $G_F(0,r) \neq 0$ :

$$G_F(0^+,r) = i\,D(0,r) = i\int \frac{d^3\vec{p}}{(2\pi)^3\, 2p_0}\, e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{i}{(2\pi)^2}\int_0^\infty \frac{p^2\,dp}{2\sqrt{p^2+m^2}}\int_\pi^0 d\cos\theta\; e^{ipr\cos\theta}$$




$$= +\frac{1}{(2\pi)^2 r}\int_0^\infty \frac{p\,dp}{2\sqrt{p^2+m^2}}\left(e^{ipr} - e^{-ipr}\right)$$

$$= +\frac{1}{2(2\pi)^2 r}\int_{-\infty}^{+\infty} \frac{p\,dp}{\sqrt{p^2+m^2}}\, e^{ipr}$$

pas de pôle → intégrale sur le contour fermé $= 0$ (on ferme en haut car $R\,e^{iRr} \xrightarrow[R\to+i\infty]{} 0$)

⇒ discontinuité : $\left(\sqrt{e^{i\pi}}\right)^{-1} - \left(\sqrt{e^{-i\pi}}\right)^{-1} = e^{-\frac{i\pi}{2}} - e^{+\frac{i\pi}{2}} = -2i$

$p = ip'$

donc $G_F(0^+,r) = \dfrac{1}{2(2\pi)^2 r}(+2i)\displaystyle\int_{-\infty} \frac{-p'\,dp'}{\sqrt{p'^2-m^2}}\, e^{-p'r}$

$$= \frac{i}{(2\pi)^2 r}\int_m^\infty dp\, \frac{p}{\sqrt{p^2-m^2}}\, e^{-pr}$$

$r$ grand : méthode du col

intégrale dominée par $p \sim m$

$$\frac{p}{\sqrt{p^2-m^2}} \underset{p\sim m}{\sim} \frac{1}{\sqrt{p-m}}\sqrt{\frac{m}{2}} \qquad X = \sqrt{p-m} \qquad 2X\,dX = dp$$

$$G_F(0^+,r) \sim \frac{i}{(2\pi)^2 r}\sqrt{\frac{m}{2}}\int_0^{+\infty}\frac{2x}{x}\,dx\, e^{-x^2 r}\, e^{-mr}$$

$$\sim \frac{i}{(2\pi)^2 r}\sqrt{\frac{m}{2}}\sqrt{\frac{\pi}{r}}\, e^{-mr}$$

$$\| \; G_F(0^+,r) \underset{r\to\infty}{\sim} \frac{i\,e^{-mr}}{(2\pi)^2 r^2}\left(\frac{\pi m r}{2}\right)^{1/2}$$

Boson de spin 1:

$$\Box A^\mu - \partial^\mu(\partial A) = j^\mu$$

T.F $\hookrightarrow$ $-p^2 A^\mu + p^\mu p_\nu A^\nu = \tilde{j}^\mu$

soit: $L_{1\,\nu}^{\mu} A^\nu = -\tilde{j}^\mu$ avec $L_{1\nu}^{\mu} = p^2 g^\mu_\nu - p^\mu p_\nu$

$L_{1\nu}^{\mu}\, p^\nu = 0$ donc $\det L_1 = 0$ $\Rightarrow$ $L_1$ n'a pas d'inverse

rem: $L_{1\,\rho}^{2\mu} = L_{1\,\nu}^{\mu} L_{1\,\rho}^{\nu} = (p^2 g^\mu_\nu - p^\mu p_\nu)(p^2 g^\nu_\rho - p^\nu p_\rho)$

$$= p^4 g^\mu_\rho - p^2 p^\mu p_\rho - p^\mu p_\rho p^2 + p^2 p^\mu p_\rho$$

$$= p^2 (p^2 g^\mu_\rho - p^\mu p_\rho)$$

donc $\| \; L_1^2 = p^2 L_1$ $\quad L_1^2$ est proportionnel à un projecteur

6°) $\mathcal{L}_{e.m} = -\frac{1}{4}F^2 - jA - \frac{\lambda}{2}(\partial A)^2$

$$\mathcal{L}_1 = -\frac{1}{4}F^2 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\frac{\delta \mathcal{L}_1}{\delta \partial_\mu A_\nu} = -\frac{4}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -F^{\mu\nu}$$

Euler Lagrange: $\frac{\delta \mathcal{L}}{\delta A_\nu} - \partial_\mu \frac{\delta \mathcal{L}_{em}}{\delta \partial_\mu A_\nu} = 0$ donne ici:

$$\partial^2 A^\nu - \partial^\nu(\partial . A) - j^\nu + \lambda \partial^\nu(\partial . A) = 0$$

soit: $\left[\Box g^\mu_\nu - (1-\lambda)\partial^\mu \partial_\nu\right] A^\nu = j^\nu$

$$M_{\mu\nu}(p) = p^2 g_{\mu\nu} - (1-\lambda) p_\mu p_\nu \qquad (= L_{1\mu\nu} \text{ pour } \lambda = 0)$$

$$L_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}$$ projecteur le long du vecteur $p$

$$T_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$$ projecteur sur les directions $\perp$ à $p$

$$L_{\mu\rho} L^\rho{}_\nu = \frac{p_\mu p_\rho}{p^2} \frac{p^\rho p_\nu}{p^2} = \frac{p_\mu p_\nu}{p^2} = L_{\mu\nu}$$

$$M_{\mu\nu} = p^2 (L+T)_{\mu\nu} - p^2 (1-\lambda) L_{\mu\nu}$$
$$= L_{\mu\nu} \lambda p^2 + T_{\mu\nu} p^2$$

rem: $L.T = 0$ car $L_{\mu\rho} T^\rho{}_\nu = 0$

$$(M^{-1})_{\mu\nu} = \frac{1}{\lambda} L_{\mu\nu} \cdot \frac{1}{p^2 + i\varepsilon} + T_{\mu\nu} \frac{1}{p^2 + i\varepsilon}$$ (prescription de Feynman)

$$= \frac{1}{p^2 + i\varepsilon} \left( \frac{1}{\lambda} \frac{p_\mu p_\nu}{p^2} + g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right)$$

$$= \frac{1}{p^2 + i\varepsilon} \left( g^{\mu\nu} + \frac{1-\lambda}{\lambda} \frac{p^\mu p^\nu}{p^2} \right)$$

$$G_{F\mu\nu}(x-y, \lambda) = -\frac{1}{(2\pi)^4} \int d^4 p \; e^{-ip(x-y)} \frac{g^{\mu\nu} + \frac{1-\lambda}{\lambda} \frac{p^\mu p^\nu}{p^2}}{p^2 + i\varepsilon}$$

$\lambda = 1$: propagateur de Feynman $\dfrac{g^{\mu\nu}}{p^2 + i\varepsilon}$ :

$$G_{F\mu\nu}(x-y, 1) = g^{\mu\nu} G_F(x-y)$$

## 2- Radiation

### 2-1 Champ créé par une charge ponctuelle

1°) $$j^\mu(t, \vec{y}) = e\,\delta^3(\vec{y} - \vec{x}(t)) \frac{dx^\mu}{dt}$$

$$= e \int_{-\infty}^{+\infty} \delta(t-s)\, ds\; \delta^3(\vec{y} - x(s)) \frac{dx^\mu}{dt}$$

$$= e \int^{+\infty} ds \frac{dx^\mu}{} \delta^4(y - x(s)) \qquad x^0(s) = s$$

8°) $$A^\mu(\bar{x}) = \frac{e}{2\pi}\int_{-\infty}^{+\infty} ds\, d^4x\, \frac{dx^\mu}{ds}\, \delta^4(x - x(s))\, \theta(\bar{x}^0 - x^0)\, \delta((\bar{x}-x)^2)$$

$$= \frac{e}{2\pi}\int_{-\infty}^{+\infty} ds\, \frac{dx^\mu}{ds}\, \theta(\bar{x}^0 - x^0(s))\, \delta((\bar{x} - x(s))^2)$$

Soit $g(s)$ une fonction de $s$

Calculons $$I_1 \equiv \int_{-\infty}^{+\infty} ds\, g(s)\, \theta(\bar{x}^0 - x^0(s))\, \delta((\bar{x} - x(s))^2)$$

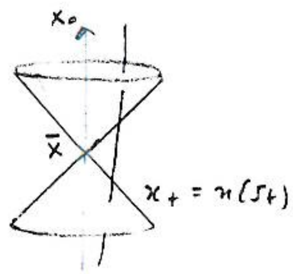
$$\delta((y - x(s))^2) = \frac{1}{\bar{x}^0 - x^0(s) + \sqrt{(\vec{\bar{x}} - \vec{x}(s))^2}}\, \delta\left(\bar{x}^0 - x^0(s) - \sqrt{(\vec{\bar{x}} - \vec{x}(s))^2}\right)$$

$$+ \frac{1}{\bar{x}^0 - x^0(s) - \sqrt{(\vec{\bar{x}} - \vec{x}(s))^2}}\, \delta\left(\bar{x}^0 - x^0(s) + \sqrt{(\vec{\bar{x}} - \vec{x}(s))^2}\right)$$

Donc $$I_1 = \int_{-\infty}^{+\infty} ds\, g(s)\, \frac{1}{\bar{x}^0 - x^0(s) + \sqrt{(\vec{\bar{x}} - \vec{x}(s))^2}}\, \delta\left(\bar{x}^0 - x^0(s) - \sqrt{(\vec{\bar{x}} - \vec{x}(s))^2}\right)$$

$$I_2 \equiv \int_{-\infty}^{+\infty} ds\, g(s)\, \frac{1}{\bar{x}^0 - x^0(s) + \sqrt{(\vec{\bar{x}} - \vec{x}(s))^2}}\, \frac{1}{|f'(s_+)|}\, \delta(s - s_+)$$

$$f'(s) = -\frac{dx^0(s)}{ds} + \frac{d\vec{x}}{ds}\, \frac{\vec{\bar{x}} - \vec{x}}{\sqrt{(\vec{\bar{x}} - \vec{x}(s))^2}}$$




$f(s_+) = 0$

$\frac{dx^\mu}{ds}$ de genre temps donc $f'(s)$ négatif

donc : $$\left(\bar{x}^0 - x^0(s_+) + \sqrt{(\vec{\bar{x}} - \vec{x}(s_+))^2}\right) |f'(s_+)|$$

$$= 2\, \frac{dx_+}{ds} \cdot (\bar{x} - x_+)$$

d'où $$I_1 = \frac{g(s_+)}{2\, \frac{dx_+}{ds} \cdot (\bar{x} - x_+)}$$

avec $g(s) = \frac{dx^\mu}{ds}$.

$$A^\mu(\bar{x}) = \frac{e}{4\pi}\, \frac{dx_+^\mu}{ds}\, \frac{1}{\frac{dx_+}{ds} \cdot (\bar{x} - x_+)} = \frac{e}{4\pi}\left(\frac{dx^\mu}{ds}\, \frac{1}{\frac{dx}{ds} \cdot (\bar{x} - x)}\right)\Bigg|_{ret}$$

"ret" = évaluation au temps $s_+$

Soit encore : $A^\mu(y) = \frac{e\, \dot{x}_+^\mu}{\ldots}$ avec $\dot{\ } = d$

réf. au repos : $\quad \dot{x}^\mu_+ = (1,0,0,0)$

donc $\begin{cases} A^0(\bar{x}) = \dfrac{e}{4\pi\,|\vec{x}-\vec{x}|} \\ \vec{A}(\bar{x}) = \vec{0} \end{cases}$

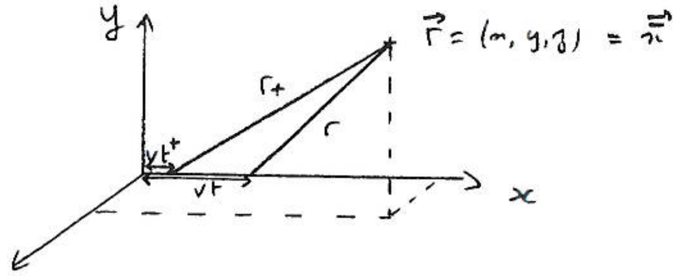
potentiel de Coulomb

9°) lien avec les transformations de Lorentz :

considérons le quadripotentiel vecteur créé en un point $\bar{x} = (t, \vec{r})$ avec $\vec{r} = (x, y, z)$ par une charge en translation uniforme, de vitesse $v$, selon l'axe des $x$

$$x(s) = (t, vt, 0, 0) = \frac{1}{\sqrt{1-v^2}}(s, vs, 0, 0)$$

$t=0$ : charge en $O$ $(0,0,0)$
$t$ : " " $(vt, 0, 0)$




$r$ : distance entre la position de la charge au temps $t$ et le point d'espace où nous voulons évaluer le potentiel

vitesse finie de propagation des interactions

$\Rightarrow$ c'est l'effet de la charge au temps $t_+ = \dfrac{1}{\sqrt{1-v^2}}\, s_+$ qui compte, avec $\quad t - t_+ = \sqrt{(x - vt_+)^2 + y^2 + z^2} = r_+$

soit $\vec{r}_+ = \vec{x} - \vec{x}_+ = (x - vt_+, y, z)$

$t - t_+ = r_+ \qquad r_+^2 = (x - vt_+)^2 + y^2 + z^2 = (x + v(t - t_+) - vt)^2 + y^2 + z^2$

$$r_+^2(1 - v^2) - 2(x - vt)v\, r_+ - (x - vt)^2 - y^2 - z^2 = 0$$

$\Delta' : (x - vt)^2 v^2 + (1 - v^2)\left((x - vt)^2 + y^2 + z^2\right)$

$$r_+ = t - t_+ = \frac{v}{1-v^2}(x - vt) + \frac{1}{1-v^2}\sqrt{(x - vt)^2 + (1 - v^2)(y^2 + z^2)}$$

$$\frac{dx_+^0}{dt_+} = 1 \qquad \frac{d\vec{x}_+}{dt_+} = \vec{v}_+$$

donc $\displaystyle \frac{dx}{dt}\cdot(\tilde{x}-x)\Big|_{ret} = (t-t_+) - \vec{v}\cdot\vec{r}_+ = r_+ - \vec{v}\cdot\vec{r}_+$

donc
$$\begin{cases} A^0_{ret}(t,\vec{r}) = \dfrac{e}{4\pi}\dfrac{1}{r_+ - \vec{v}\cdot\vec{r}_+} \\[2ex] \vec{A}_{ret}(t,\vec{r}) = \dfrac{e}{4\pi}\dfrac{\vec{v}_+}{r_+ - \vec{v}\cdot\vec{r}_+} \end{cases}$$

ou encore:
$$\begin{cases} A^0_{ret}(t,x,y,z) = \dfrac{1}{4\pi}\dfrac{e}{\sqrt{1-v^2}}\dfrac{1}{\sqrt{\left(\frac{x-vt}{\sqrt{1-v^2}}\right)^2 + y^2 + z^2}} \\[2ex] \vec{A}_{ret}(t,x,y,z) = \vec{v}\,A^0 \end{cases}$$

en effet
$$\begin{aligned} r_+ - \vec{v}\cdot\vec{r}_+ &= r_+ - vx + v^2 t_+ = r_+ - vx + v^2(t - r_+) \\ &= r_+(1-v^2) + v^2 t - vx \\ &= v(x-vt) + v^2 t - vx + \sqrt{(x-vt)^2 + (1-v^2)(y^2+z^2)} \end{aligned}$$

$$r_+ - \vec{v}\cdot\vec{r}_+ = \sqrt{(x-vt)^2 + (1-v^2)(y^2+z^2)}$$

rem: $\displaystyle \frac{\frac{dx^\mu}{ds}}{\frac{dx^\mu}{ds}\cdot(\tilde{x}-x)}\Bigg|_{ret} = \frac{\frac{dx^\mu}{dt}}{\frac{dx^\mu}{dt}\cdot(\tilde{x}-x)}\Bigg|_{ret} \qquad \left(\frac{dt}{ds} = \frac{1}{\sqrt{1-v^2}}\right)$

dans le référentiel où la particule est au repos:

$A = A_{Coulomb}$ : $\begin{cases} A'^0 = \dfrac{e}{4\pi}\dfrac{1}{\sqrt{x'^2+y'^2+z'^2}} \\ \vec{A}' = 0 \end{cases}$

boost: $\begin{cases} t' = \dfrac{t - vx}{\sqrt{1-v^2}} \\ x' = \dfrac{x - vt}{\sqrt{1-v^2}} \\ y' = y \\ z' = z \end{cases}$ donc $\begin{cases} A'^0 = \dfrac{A^0 - \vec{v}\cdot\vec{A}}{\sqrt{1-v^2}} \\ \vec{A}' = \vec{0} = \vec{A} - \vec{v}A^0 \end{cases}$ $\begin{cases} A^0 = \dfrac{1}{\sqrt{1-v^2}}A'^0 \\ \vec{A} = \vec{v}A^0 \end{cases}$

d'où $\begin{cases} A^0 = \dfrac{e}{4\pi} \dfrac{1}{\sqrt{1-v^2}} \dfrac{1}{\sqrt{\left(\dfrac{x - vt}{\sqrt{1-v^2}}\right)^2 + y^2 + z^2}} \\ \vec{A} = \vec{v} A^0 \end{cases}$

## Calcul de $\vec{E}$ et $\vec{B}$.

$$E^i = F^{i0}$$

$$B_i = -\frac{1}{2} \varepsilon_{ijk} F^{jk}$$

$$A^\nu(\bar{x}) = \frac{e}{2\pi} \int_{-\infty}^{+\infty} ds \, \frac{dx^\nu}{ds} \, \Theta(\bar{x}^0 - x^0(s)) \, \delta((\bar{x} - x(s))^2)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial^\mu A^\nu = \frac{e}{2\pi} \int ds \, \frac{dx^\nu}{ds} \, \Theta(\bar{x}^0 - x^0(s)) \, \partial^\mu \delta((\bar{x} - x(s))^2) \qquad \text{pour } r_+ \neq 0$$

(la dérivée de $\Theta(\bar{x}^0 - x^0(s))$ donnerait, avec $\delta((\bar{x} - x(s))^2)$, $\delta(|\vec{x} - \vec{x}(s)|^2) = \delta(r_+^2)$).

$$\partial^\mu \delta(F(s)) = (\partial_\mu F) \frac{d}{dF} \delta(F(s)) = \partial_\mu F \, \frac{ds}{dF} \, \frac{d}{ds} \delta(F(s))$$

avec $F(s) = (\bar{x} - x(s))^2$

$$\frac{dF}{ds} = -2 \frac{dx}{ds} \cdot (\bar{x} - x(s)) \qquad \partial^\mu F = 2(\bar{x}^\mu - x^\mu(s))$$

$$\partial^\mu \delta(|\bar{x} - \vec{x}(s)|^2) = -\frac{\bar{x}^\mu - x^\mu(s)}{V \cdot (\bar{x} - x(s))} \frac{d}{ds} \delta((\bar{x} - x(s))^2)$$

avec $V = \dfrac{d x(s)}{ds}$

par intégration par partie:

$$\partial^\mu A^\nu = \frac{e}{2\pi} \int ds \, \frac{d}{ds} \left[ \frac{V^\nu (\bar{x}^\mu - x^\mu(s))}{V \cdot (\bar{x} - x(s))} \right. \Theta(\bar{x}^0 - x^0(s)) \, \delta((\bar{x} - x(s))^2)$$

(la différentiation de $\Theta(\bar{x}^0 - x^0(s))$ donnerait à nouveau un $\delta(r_+)$).

d'où $$F^{\mu\nu} = \frac{e}{2\pi}\int_{-\infty}^{+\infty} ds \, \frac{d}{ds}\left[\frac{V^\nu(\bar{x}^\mu - x^\mu(s)) - V^\mu(\bar{x}^\nu - x^\nu(s))}{V.(\bar{x} - x(s))}\right]\theta(\bar{x}^0 - x^0(s))$$
$$\times \delta((\bar{x} - x(s))^2)$$

$$F^{\mu\nu} = \frac{e}{4\pi}\,\frac{1}{V\cdot(\bar{x}-x)}\,\frac{d}{ds}\,\frac{V^\nu(\bar{x}^\mu - x^\mu(s)) - V^\mu(\bar{x}^\nu - x^\nu(s))}{V.(\bar{x} - x)}\Bigg|_{ret}$$

$$V_+ = \frac{dx_+}{ds}$$

(i.e évalué en $s = s_+$)

$$V^\nu \frac{d}{ds}(\bar{x}^\mu - x^\mu(s)) - V^\mu \frac{d}{ds}(\bar{x}^\nu - x^\nu(s)) = 0$$

$$F^{\mu\nu} = \frac{e}{4\pi}\,\frac{\dot{V}^\nu(\bar{x}^\mu - x^\mu) - \dot{V}^\mu(\bar{x}^\nu - x^\nu)}{(V.(\bar{x} - x))^2}$$

$\boxed{\cdot = \frac{d}{ds}}$

$$- \frac{e}{4\pi}\,\frac{1}{(V.(\bar{x}-x))^3}\,\dot{V}.(\bar{x}-x)\left[V^\nu(\bar{x}^\mu - x^\mu) - V^\mu(\bar{x}^\nu - x^\nu)\right]$$

$$+ \frac{e}{4\pi}\,\frac{1}{(V.(\bar{x}-x))^3}\,\underset{\underset{1}{\shortparallel}}{V^2}\left[V^\nu(\bar{x}^\mu - x^\mu) - V^\mu(\bar{x}^\nu - x^\nu)\right]$$

(dans le réf. au repos de la particule, $V^\mu = (1, \vec{0})$)

$$V.(\bar{x} - x)|_{ret} = \frac{r_+ - \vec{v}.\vec{r}_+}{\sqrt{1 - v^2}}$$

posons $\vec{n} = \dfrac{\bar{x} - x_+}{r_+} = \dfrac{\vec{r}_+}{r_+}$ et $n^\mu = (1, \vec{n})$

$$F^{\mu\nu} = \frac{e}{4\pi r_+^2}\,\frac{1}{(V.n)^3}\left[V^\nu n^\mu - V^\mu n^\nu\right]$$
$$+ \frac{e}{4\pi r_+}\,\frac{1}{(V.n)^3}\left[(V.n)(\dot{V}^\nu n^\mu - \dot{V}^\mu n^\nu) - (\dot{V}.n)(V^\nu n^\mu - V^\mu n^\nu)\right]$$

1^er terme : champ statique (ne dépend pas de l'accélération) décroit en $\frac{1}{r_+^2}$

2ième terme : dépend de l'accélération et décroit en $\frac{1}{r_+}$ (champ de radiation)

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$$

$$\tilde{F}_{\mu\nu} n^\nu = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} n^\nu$$

or $F^{\alpha\beta}$ s'écrit $F^{\alpha\beta} = A^\alpha n^\beta - A^\beta n^\alpha$

donc $\tilde{F}_{\mu\nu} n^\nu = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}(A^\alpha n^\beta - A^\beta n^\alpha) n^\nu$

$$= \frac{1}{2}\underbrace{\varepsilon_{\mu\alpha\beta\nu}}_{\varepsilon_{\mu\nu\alpha\beta}} A^\beta n^\nu n^\alpha - \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} A^\beta n^\alpha n^\nu = 0$$

$\tilde{F}_{i0} + \tilde{F}_{ij} n^j = 0$ or $\begin{cases} E^i = F^{i0} \\ B_i = -\frac{1}{2}\varepsilon_{ijk} F^{jk} \end{cases}$

$$\tilde{F}_{ij} = \frac{1}{2}\varepsilon_{ijk0} F^{k0} + \frac{1}{2}\varepsilon_{ij0k} F^{0k}$$

$$= \varepsilon_{ijk0} F^{k0} = \varepsilon_{ijk} F^{k0} \quad \text{car } \varepsilon_{ijk0} = \varepsilon_{ijk}$$

$$\tilde{F}_{i0} = \frac{1}{2}\varepsilon_{i0\alpha\beta} F^{\alpha\beta} = \frac{1}{2}\varepsilon_{i\alpha\beta 0} F^{\alpha\beta} = \frac{1}{2}\varepsilon_{ijk} F^{jk} = -B_i$$

d'où $B_i = \tilde{F}_{ij} n^j = \varepsilon_{ijk} n^j E^k = (\vec{n} \wedge \vec{E})_i$

$$\Downarrow \quad \vec{B} = \vec{n} \wedge \vec{E}$$

<u>champ $\vec{E}$ :</u> $E^i = F^{i0}$

$$\frac{dV^\alpha}{d\tau} = \left(\gamma^4 \vec{v}\cdot\dot{\vec{v}},\ \gamma^2 \dot{\vec{v}} + \gamma^4 \vec{v}(\vec{v}\cdot\dot{\vec{v}})\right) \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$V\cdot(\tilde{x} - x)\big|_{ret} = \gamma(r_+ - \vec{v}\,\vec{r}_+)$$

$$\tilde{x} - x_+ = r_+ = (r_+, \vec{r}_+) \qquad V = (\gamma, \gamma\vec{v})$$

<u>donc</u> : $E^i = \frac{e}{4\pi}\frac{1}{(r_+ - \vec{r}_+\cdot\vec{v}_+)^3}\frac{1}{\gamma^3}\Big[ (V_+^0 r_+^i - V_+^i r_+)$

$$+ (\dot{V}_+^0 r_+^i - \dot{V}_+^i r_+^0)(r_+ - \vec{r}_+\cdot\vec{v}_+)\gamma - \underbrace{(\dot{V}\cdot r_+)}_{\gamma(r_+ - \vec{v}\cdot\vec{r}_+)}(V^0 r_+^i - V^i r_+^0)\Big]$$

$$\vec{E} = \frac{e}{4\pi} \frac{1}{(r_t - \vec{r}_t \cdot \vec{v}_t)^3} \Big[ (\vec{r}_t - \vec{v}_t\, r_t)\, \gamma^{-2}$$

$$+ \left(\gamma^2 (\vec{v}_t \cdot \dot{\vec{v}}_t)\, \vec{r}_t - r_t\, \dot{\vec{v}}_t - \gamma^2\, \vec{v}_t\, (\vec{v}_t \cdot \dot{\vec{v}}_t)\, r_t\right) (r_t - \vec{r}_t \cdot \vec{v}_t)$$

$$- \left(\gamma^2\, \vec{v}_t \cdot \dot{\vec{v}}_t\, r_t - \dot{\vec{v}}_t \cdot \vec{r}_t - \gamma^2\, \vec{v}_t \cdot \vec{r}_t\, (\vec{v}_t \cdot \dot{\vec{v}}_t)\right) (\vec{r}_t - \vec{v}_t\, r_t) \Big]$$

$$= \frac{e}{4\pi} \frac{1}{(r_t - \vec{r}_t \cdot \vec{v}_t)^3} \Big[ (\vec{r}_t - \vec{v}_t\, r_t)(1 - \vec{v}^2) - r_t\, \dot{\vec{v}}_t\, (r_t - \vec{r}_t \cdot \vec{v}_t)$$

$$+ (\vec{r}_t \cdot \dot{\vec{v}})(\vec{r}_t - \vec{v}_t\, r_t) \Big]$$

or $\vec{r}_t \wedge \left( (\vec{r}_t - r_t \vec{v}_t) \wedge \dot{\vec{v}}_t \right) = \vec{r}_t \cdot \dot{\vec{v}}_t\, (\vec{r}_t - r_t \vec{v}_t) - \left( \vec{r}_t \cdot (\vec{r}_t - r_t \vec{v}_t) \right) \dot{\vec{v}}_t$

$$\vec{r}_t\, (\vec{r}_t - r_t \vec{v}) = r_t^2 - r_t\, \vec{r}_t \cdot \vec{v} = r_t\, (r_t - \vec{r}_t \cdot \vec{v}_t)$$

done:

$$\boxed{\vec{E} = \frac{e}{4\pi} \frac{1}{r - \vec{r} \cdot \vec{v}} \Big[ (\vec{r} - r\vec{v})(1 - \vec{v}^2) + \vec{r} \wedge \big[ (\vec{r} - r\vec{v}) \wedge \dot{\vec{v}} \big] \Big] \Big|_{ret}}$$

## 2-2 Bremsstrahlung classique

10°) $$x(\tau) = \begin{cases} \frac{p_i}{m}\,\tau & \tau < 0 \\ \frac{p_f}{m}\,\tau \end{cases}$$

$$j_\mu(x) = e\int d\tau\, \frac{dx_\mu}{d\tau}\,\delta^4(x - x(\tau))$$

$$= e\int_{-\infty}^{0} d\tau\, \frac{dx_\mu}{d\tau}\,\delta^4\left(x - \frac{p_i}{m}\tau\right) + \int_0^{+\infty} d\tau\, \frac{dx_\mu}{d\tau}\,\delta^4\left(x - \frac{p_f}{m}\tau\right)$$

$$= e\int \frac{d^4k}{(2\pi)^4}\left[\int_{-\infty}^{0} e^{-ik\left(x - \frac{p_i}{m}\tau\right)} + \int_0^{+\infty} e^{-ik\left(x - \frac{p_f}{m}\tau\right)}\right] d\tau\, \frac{dx_\mu}{d\tau}$$

$$= e\int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\left[\int_{-\infty}^{0} e^{i\frac{k.p_i}{m}\tau}\,\frac{p^i_\mu}{m}\,d\tau + \int_0^{+\infty} e^{i\frac{k.p_f}{m}\tau}\,\frac{p^f_\mu}{m}\,d\tau\right]$$

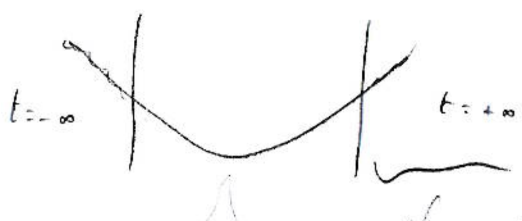
$$= e\int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\left\{\left[\frac{m}{ikp_i}\,\frac{p^i_\mu}{m}\,e^{i\frac{k.p_i}{m}\tau}\right]_{-\infty}^{0} + \left[\frac{m}{ikp_f}\,\frac{p^f_\mu}{m}\,e^{i\frac{k.p_f}{m}\tau}\right]_0^{+\infty}\right\}$$

$$= \frac{-ie}{(2\pi)^4}\int d^4k\, e^{-ikx}\left[\frac{p^i_\mu}{p^i.k} - \frac{p^f_\mu}{p^f.k}\right]$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\,\tilde{j}_\mu(k) \qquad \tilde{j}_\mu(k) = -ie\left[\frac{p^i_\mu}{p^i.k} - \frac{p^f_\mu}{p^f.k}\right]$$

11°) $$A^\mu(x) = \int d^4x'\, G_{ret}(x - x')\, j^\mu(x')$$

$$= \underbrace{\int d^4x'\, G^{(-)}(x - x')\, j^\mu(x')}_{\substack{\text{Champ de radiation} \\ \text{solution de l'éq. de Maxwell} \\ \text{homogène } \Box A^\mu = 0}} + \underbrace{\int d^4x'\, G_{av}(x - x')\, j^\mu(x')}_{\substack{\text{Champ de Coulomb} \\ \text{attaché à la particule}}}$$



12°) $A^{\mu}_{rad}(x) = \int d^4x' \, G^{(-)}(x-x') \, j^{\mu}(x')$

$$= \frac{i}{(2\pi)^3} \int d^4k \; e^{-ikx} \; \varepsilon(k_0) \, \delta(k^2) \, \tilde{j}^{\mu}(k)$$

$$F^{\mu\nu}_{rad}(x) = \partial^{\mu} A^{\nu}_{rad} - \partial^{\nu} A^{\mu}_{rad}$$

$$= \frac{i}{(2\pi)^3} \int d^4k \; e^{-ikx} \; \varepsilon(k_0) \, \delta(k^2) \left( -ik^{\mu} \tilde{j}^{\nu}(k) + ik^{\nu} \tilde{j}^{\mu}(k) \right)$$

$$= \frac{1}{(2\pi)^3} \int d^4k \; e^{-ikx} \; \varepsilon(k_0) \, \delta(k^2) \left[ k^{\mu} \tilde{j}^{\nu}(k) - k^{\nu} \tilde{j}^{\mu}(k) \right]$$

$$\mathcal{E} = \int d^3\vec{x} \; \Theta^{00}(t, \vec{x}) = \frac{1}{(2\pi)^6} \int d^3\vec{x} \int d^4k \, d^4k' \; e^{-i(k+k')x} \, \varepsilon(k_0)\varepsilon(k_0')\delta(k^2)\delta(k'^2)$$

$$\left.\begin{aligned} \times & \Big[ \tfrac{1}{4} \left(k^{\mu} \tilde{j}^{\nu}(k) - k^{\nu} \tilde{j}^{\mu}(k)\right) \left(k'_{\mu} \tilde{j}_{\nu}(k') - k'_{\nu} \tilde{j}_{\mu}(k')\right) \\ & + \left(k^0 \tilde{j}^{\rho}(k) - k^{\rho} \tilde{j}^0(k)\right)\left(k'_{\rho} \tilde{j}^0(k') - k'^0 \tilde{j}_{\rho}(k')\right) \Big] \end{aligned}\right\} T(k,k')$$

$$\delta(k^2) = \frac{1}{2k_0} \delta(|\vec{k}| - k_0) - \frac{1}{2k_0} \delta(|\vec{k}| + k_0)$$

$$\varepsilon(k_0) \, \delta(k^2) = \frac{1}{2k_0} \delta(|\vec{k}| - k_0) + \frac{1}{2k_0} \delta(|\vec{k}| + k_0)$$

$$\frac{1}{(2\pi)^3} \int d^3\vec{x} \; e^{-i(k+k')x} = \delta^3(\vec{k} + \vec{k}') \; e^{-i(k_0 + k_0')t_0}$$

Donc : $\mathcal{E} = \frac{1}{(2\pi)^6} \int d^3x \, d^4k \, d^4k' \; e^{-i(k+k')x} \; \varepsilon(k_0)\varepsilon(k_0')\delta(k^2)\delta(k'^2) \; T(k,k')$

$$= \frac{1}{(2\pi)^3} \int \frac{dk_0 \, dk_0'}{4k_0 k_0'} \, d^3k \, d^3k' \; e^{-i(k_0+k_0')t_0} \left(\delta(|\vec{k}| + k_0) + \delta(|\vec{k}| - k_0)\right)\left(\delta(|\vec{k}'| + k_0') + \delta(|\vec{k}'| - k_0')\right)$$

$$\times \, \delta(\vec{k} + \vec{k}') \; T(k,k')$$

$T(k,k')$ est symétrique dans les transformations $k \to -k$ et $k' \to -k'$ ($\tilde{j}(-k) = -\tilde{j}(k) = \tilde{j}^*(k)$).

* terme $\delta(|\vec{k}| + k_0)\,\delta(|\vec{k}'| - k_0')\,\delta(\vec{k} + \vec{k}')$ : on obtient $|\vec{k}'| = -|\vec{k}| = k_0' = -k_0$

soit $k = -k'$

$$T(k,k') = \tfrac{1}{2}\left( k.k' \, \tilde{j}(k).\tilde{j}(k') - k.\tilde{j}(k') \, k'.\tilde{j}(k) \right) - k^0 k'^0 \, \tilde{j}(k).\tilde{j}(k')$$

$$+ \, k^0 \tilde{j}^0(k') \, k'.\tilde{j}(k) + k'^0 \tilde{j}^0(k) \, k.\tilde{j}(k') - k.k' \, \tilde{j}^0(k) \tilde{j}^0(k')$$

Par conservation du courant, $k.\tilde{j}(k)=0$

ici, comme $k=-k'$ et $k^2=0$, le seul terme qui est non nul est $-k^0 k'^0\, \tilde{j}(k).\tilde{j}(k') = k^{0^2}\, \tilde{j}(k).\tilde{j}(-k)$

$k.\tilde{j}(k)=0$ donc $\tilde{j}(k)$ peut se décomposer sur $(k, \varepsilon_1, \varepsilon_2)$

avec $\varepsilon_1^2 = \varepsilon_2^2 = -1 \quad \varepsilon_1.\varepsilon_2 = 0 \quad k.\varepsilon_\lambda = 0$

$$\tilde{j}(k).\tilde{j}(-k) = \tilde{j}(k).\tilde{j}^*(k) = \left[\sum_\lambda (\tilde{j}(k).\varepsilon_\lambda)\varepsilon_\lambda\right].\left[\sum_{\lambda'} (\tilde{j}(k).\varepsilon_{\lambda'})\varepsilon_{\lambda'}\right]$$

$(k^2=0$ et $k.\varepsilon_\lambda = 0)$

$$= -\sum_\lambda |\varepsilon_\lambda.\tilde{j}(k)|^2 \quad (\text{car } \varepsilon_\lambda^2 = -1)$$

* <u>terme</u> $\delta(|\vec{k}|-k_0)\,\delta(|\vec{k}'|+k_0')\,\delta(\vec{k}+\vec{k}')$ : même contribution

ces deux termes donnent donc $2.\dfrac{1}{(2\pi)^2}\,\dfrac{1}{4}\displaystyle\int d^3k \sum_\lambda |\varepsilon_\lambda.\tilde{j}(k)|^2$

la contribution des 2 autres termes est nulle :

<u>ex</u>: $\dfrac{d^4k\, d^4k'}{4k_0k_0'}\;\delta(|\vec{k}|+k_0)\,\delta(|\vec{k}'|+k_0')\,\delta(\vec{k}+\vec{k}')\,e^{-i(k_0+k_0')t_0}$

$$\xrightarrow{k\to -k} \frac{d^4k\, d^4k'}{(-4k_0k_0')}\;\delta(|\vec{k}|-k_0)\,\delta(|\vec{k}'|+k_0')\,\delta(\vec{k}-\vec{k}')\,e^{-i(-k_0+k_0')t_0}$$

$$\text{donc} \left.\begin{array}{l} k_0 = |\vec{k}| = -k_0' \\ \vec{k} = -\vec{k}' \end{array}\right\}\; k = -k'$$

$T(k,k') \longrightarrow T(\tilde{k},k')$ avec $\tilde{k} = (-k^0, \vec{k})$

$$T(\tilde{k},k') = \frac{1}{2}\Big(\underset{①}{\tilde{k}.k'\,\tilde{j}(\tilde{k}).\tilde{j}(k')} - \underset{②}{\tilde{k}.\tilde{j}(k')\,k'.\tilde{j}(\tilde{k})}\Big) + \underset{③}{k^0k'^0\,\tilde{j}(\tilde{k}).\tilde{j}(k')}$$

$$- \underset{④}{k^0\tilde{j}^0(k')\,k'.\tilde{j}(\tilde{k})} + \underset{⑤}{k'^0\,\tilde{j}^0(\tilde{k})\,\tilde{k}.\tilde{j}(k')} - \underset{⑥}{\tilde{k}.k'\,\tilde{j}^0(\tilde{k})\,\tilde{j}^0(k')}$$

$\frac{1}{2}\tilde{k}.k' + k^0k'^0 = \frac{1}{2}(k^{0^2} + \vec{k}^2) - k^{0^2} = k^{0^2} - k^{0^2} = 0 \qquad (k' = -k$ donc ① + ③ $= 0$

$$-\frac{1}{2}\tilde{k}.\tilde{j}(-k) = -\frac{1}{2}\underbrace{k.\tilde{j}(-k)}_{0} + k^0\tilde{j}^0(-k) = k^0\tilde{j}^0(-k)$$

$$k'.\tilde{j}(\tilde{k}) = -k\,\tilde{j}(\tilde{k}) = -\underbrace{\tilde{k}.\tilde{j}(\tilde{k})}_{0} - 2k^0\tilde{j}^0(\tilde{k})$$

$$\text{donc}\quad \left.\begin{array}{l} ② = -2k^{0^2}\,\tilde{j}^0(-k)\,\tilde{j}^0(\tilde{k}) \\ ④ = 2k^{0^2}\,\tilde{j}^0(-k)\,\tilde{j}^0(\tilde{k}) \end{array}\right\}\; ② + ④ = 0$$

$⑤ = 2k^{0^2}\,\tilde{j}^0(\tilde{k})\,\tilde{j}^0(-k)$

[illegible] $\Big\}$ ⑤ + ⑥ = 0

finalement : $\mathcal{E} = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} \, k_0 \sum_\lambda |\varepsilon_\lambda \cdot \tilde{j}(k)|^2$

énergie émise dans un espace de phase $d^3k$ :

$$d\mathcal{E} = \frac{1}{(2\pi)^3} \frac{d^3k}{2} e^2 \sum_\lambda \left| \frac{\varepsilon_\lambda \cdot p^i}{k \cdot p^i} - \frac{\varepsilon_\lambda \cdot p^f}{k \cdot p^f} \right|^2$$

nombre de photons de polarisation $\varepsilon$ et d'énergie $k_0$ :

$$dN = \frac{d\mathcal{E}}{k^0} = e^2 \left| \frac{\varepsilon \cdot p^i}{k \cdot p^i} - \frac{\varepsilon \cdot p^f}{k \cdot p^f} \right| \frac{d^3k}{2(2\pi)^3 k^0}$$

$\rightarrow$ $\mathcal{E}$ finie

$N$ infini