Construction de Schwinger des représentations irréductibles de SU(2)

1) $J_1 = \frac{1}{2}(a_1^+ a_2 + a_2^+ a_1)$

$J_2 = \frac{i}{2}(-a_1^+ a_2 + a_2^+ a_1)$

$J_3 = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2) = \frac{1}{2}(N_1 - N_2)$

$$(J_i, J_j) = \frac{1}{4}[a^+ \sigma_i a \, , a^+ \sigma_j a]$$

$$= \frac{1}{4}\left[a^+_{\alpha'} \sigma_i^{\alpha'\beta'} a_{\beta'}, a^+_\alpha \sigma_j^{\alpha\beta} a_\beta\right]$$

$$= \frac{1}{4}\left(a^+_{\alpha'} \sigma_i^{\alpha'\beta'} a_{\beta'} a_\alpha^+ \sigma_j^{\alpha\beta} a_\beta - a^+_\alpha \sigma_j^{\alpha\beta} a_\beta a^+_{\alpha'} \sigma_i^{\alpha'\beta'} a_{\beta'}\right)$$

$$= \frac{1}{4} a^+_{\alpha'} \sigma_i^{\alpha'\beta'} \delta_{\alpha\beta'} \sigma_j^{\alpha\beta} a_\beta + \frac{1}{4} a^+_{\alpha'} \sigma_i^{\alpha'\beta'} a_\alpha^+ a_{\beta'} \sigma_j^{\alpha\beta} a_\beta$$

$$- \frac{1}{4} a^+_\alpha \sigma_j^{\alpha\beta} \delta_{\beta\alpha'} \sigma_i^{\alpha'\beta'} a_{\beta'} - \frac{1}{4} a^+_\alpha \sigma_j^{\alpha\beta} a^+_{\alpha'} a_\beta \sigma_i^{\alpha'\beta'} a_{\beta'}$$

$\to 0$ car $[a_\alpha^+, a_{\alpha'}^+] = [a_\beta, a_{\beta'}] = 0$

$$= \frac{1}{4}\left(\sigma_i^{\alpha'\alpha} \sigma_j^{\alpha\beta} a^+_{\alpha'} a_\beta - \sigma_j^{\alpha\beta} \sigma_i^{\beta\beta'} a^+_\alpha a_{\beta'}\right)$$

$$= \frac{1}{4}\left(\sigma_i^{\alpha\delta} \sigma_j^{\delta\beta} - \sigma_j^{\alpha\delta} \sigma_i^{\delta\beta}\right) a^+_\alpha a_\beta = \frac{1}{2} i \,\varepsilon_{ijk}\, \sigma_k^{\alpha\beta} a^+_\alpha a_\beta = i\varepsilon_{ijk} J_k$$

algèbre de Lie de SO(3)

2) a) $J_3 = \frac{1}{2}(N_1 - N_2)$

$$J^2 = J_1^2 + J_2^2 + J_3^2 = \frac{1}{4}(a^+_1 a_2 + a^+_2 a_1)(a^+_1 a_2 + a^+_2 a_1)$$

$$- \frac{1}{4}(-a^+_1 a_2 + a^+_2 a_1)(-a^+_1 a_2 + a^+_2 a_1) + \frac{1}{4}N_1^2 + \frac{1}{4}N_2^2 - \frac{1}{2}N_1N_2$$

$$= \frac{1}{4}\Big(\underbrace{a_1^+ a_2 a_2^+ a_1}_{N_1N_2 + N_1} + \underbrace{a_2^+ a_1 a_1^+ a_2}_{N_1N_2 + N_2} + \underbrace{a_1^+ a_2 a_2^+ a_1}_{N_1N_2 + N_1} + \underbrace{a_2^+ a_1 a_1^+ a_2}_{N_1N_2 + N_2}\Big) + \frac{1}{4}N_1^2 + \frac{1}{4}N_2^2 - \frac{1}{2}N_1N_2$$

$$= \frac{1}{2}N_1N_2 + \frac{1}{2}N_1 + \frac{1}{2}N_2 + \frac{1}{4}N_1^2 + \frac{1}{4}N_2^2 = \frac{N_1+N_2}{2}\left(\frac{N_1+N_2}{2} + 1\right)$$

Donc $J^2 |n_1 n_2\rangle = \frac{n_1+n_2}{2}\left(\frac{n_1+n_2}{2} + 1\right)|n_1 n_2\rangle$

$J_3 |n_1 n_2\rangle = \frac{n_1 - n_2}{2} |n_1 n_2\rangle$

Schw

b) $|n_1\, n_2\rangle = \dfrac{a_1^{+\,n_1}}{\sqrt{n_1!}} \dfrac{a_2^{+\,n_2}}{\sqrt{n_2!}} |0\rangle$

c) On pose $\frac{n_1+n_2}{2} = j$ et $\frac{n_1-n_2}{2} = m$, i-e $n_1 = j+m$ et $n_2 = j-m$

$n_1+n_2 = 2j$ donc j est entier ou demi-entier

$0 \leq n_1$ donc $m \geq -j$

$0 \leq n_2$ donc $j \geq m$

$$J_+ |j\, m\rangle = (J_1 + iJ_2)|j\, m\rangle = a_1^+ a_2 |n_1\, n_2\rangle$$

$$= a_1^+ \frac{a_1^{+\,n_1}}{\sqrt{n_1!}} \; a_2 \frac{a_2^{+\,n_2}}{\sqrt{n_2!}} |0\rangle$$

$$= \sqrt{n_1+1} \frac{a_1^{+\,n_1+1}}{\sqrt{(n_1+1)!}} \; n_2 \frac{a_2^{+\,n_2-1}}{\sqrt{n_2!}} |0\rangle$$

$$= \sqrt{n_2(n_1+1)}\, |n_1+1, n_2-1\rangle = \sqrt{(j-m)(j+m+1)}\, |j\; m+1\rangle$$

$$J_- |j\, m\rangle = (J_1 - iJ_2)|j\, m\rangle = a_2^+ a_1 |n_1\, n_2\rangle$$

$$= \sqrt{n_1(n_2+1)}\, |n_1-1, n_2+1\rangle = \sqrt{(j+m)(j-m+1)}\, |j\; m-1\rangle$$

3) a) $g^j_m(x_1, x_2, \theta) = \langle j\, m| \sum_{m'} \dfrac{x_1^{j+m'}}{\sqrt{(j+m')!}} \dfrac{x_2^{j-m'}}{\sqrt{(j-m')!}} \, e^{-i\theta J_2} \dfrac{a_1^{+\,j+m'}}{\sqrt{(j+m')!}} \dfrac{a_2^{+\,j-m'}}{\sqrt{(j-m')!}} |0\rangle$

$$= \langle j\, m| e^{-i\theta J_2} \sum_{m'} \frac{x_1^{j+m'}}{(j+m')!} \frac{x_2^{j-m'}}{(j-m')!} a_1^{+\,j+m'} a_2^{+\,j-m'} |0\rangle$$

$$= \langle j\, m| e^{-i\theta J_2} \frac{(x_1 a_1^+ + x_2 a_2^+)^{2j}}{(2j)!} |0\rangle$$

b) $e^{-i\theta J_2}(x_1 a_1^+ + x_2 a_2^+)^{2j}$

$$= \left[e^{-i\theta J_2}(x_1 a_1^+ + x_2 a_2^+) e^{i\theta J_2}\right]^{2j} e^{-i\theta J_2}$$

Or $e^{-i\theta J_2}|0\rangle = |0\rangle$

On pose $\alpha(\theta) = e^{-i\theta J_2}(x_1 a_1^+ + x_2 a_2^+) e^{i\theta J_2} = x_1 u + x_2 v$

$$\frac{d\alpha}{d\theta} = i e^{-i\theta J_2} [x_1 a_1^+ + x_2 a_2^+, J_2]\, e^{i\theta J_2}$$

Schw-3

$$\frac{du}{d\theta} = i e^{-i\theta J_2} [a_1^+, J_2] e^{i\theta J_2} \qquad \frac{dv}{d\theta} = i e^{-i\theta J_2} [a_2^+, J_2] e^{i\theta J_2}$$

$$[a_1^+, J_2] = -\frac{i}{2} a_2^+$$

$$[a_2^+, J_2] = \frac{i}{2} a_1^+$$

donc $\begin{cases} \frac{du}{d\theta} = \frac{v}{2} \\ \frac{dv}{d\theta} = -\frac{u}{2} \end{cases}$ $\quad W = \begin{pmatrix} u \\ v \end{pmatrix} \quad \frac{d}{d\theta} W = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W$

donc $\frac{dW}{d\theta} = \frac{i}{2} \sigma_2 W$

solution $W = e^{\frac{i}{2}\sigma_2 \theta} W_0 = \left(\cos\frac{\theta}{2} + i\sigma_2 \sin\frac{\theta}{2}\right) W_0$

$\theta = 0: \quad W_0 = \begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix}$ $\qquad$ donc $W = \begin{pmatrix} \cos\frac{\theta}{2} a_1^+ + \sin\frac{\theta}{2} a_2^+ \\ \cos\frac{\theta}{2} a_2^+ - \sin\frac{\theta}{2} a_1^+ \end{pmatrix}$

D'où $\quad e^{-i\theta J_2}(x_1 a_1^+ + x_2 a_2^+) e^{i\theta J_2} = a_1^+ \left(x_1 \cos\frac{\theta}{2} - x_2 \sin\frac{\theta}{2}\right) + a_2^+ \left(x_2 \cos\frac{\theta}{2} + x_1 \sin\frac{\theta}{2}\right)$

c) $$g^j_m (x_1, x_2; \theta) = \frac{\langle j\, m | \left[a_1^+ (x_1 \cos\frac{\theta}{2} - x_2 \sin\frac{\theta}{2}) + a_2^+ (x_2 \cos\frac{\theta}{2} + x_1 \sin\frac{\theta}{2})\right]^{2j} |0\rangle}{(2j)!}$$

$$= \sum_{m'=-j}^{j} \frac{\langle j m | a_1^{+\, j+m'} (x_1 \cos\frac{\theta}{2} - x_2 \sin\frac{\theta}{2})^{j+m'} a_2^{+\, j-m'} (x_2 \cos\frac{\theta}{2} + x_1 \sin\frac{\theta}{2})^{j-m'} |0\rangle}{(j+m')!\ (j-m')!}$$

$$= \sum_{m'} \frac{\langle j m | (x_1 \cos\frac{\theta}{2} - x_2 \sin\frac{\theta}{2})^{j+m'} (x_1 \sin\frac{\theta}{2} + x_2 \cos\frac{\theta}{2})^{j-m'} | j m' \rangle}{\sqrt{(j+m')!}\ \sqrt{(j-m')!}}$$

$$= \frac{(x_1 \cos\frac{\theta}{2} - x_2 \sin\frac{\theta}{2})^{j+m} (x_1 \sin\frac{\theta}{2} + x_2 \cos\frac{\theta}{2})^{j-m}}{\sqrt{(j+m)!(j-m)!}}$$

d) Posons $\begin{cases} x_1 = -\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ x_2 = t - \cos^2\frac{\theta}{2} \end{cases}$

$$x_1 \cos\frac{\theta}{2} - x_2 \sin\frac{\theta}{2} = -\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} - t\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\cos^2\frac{\theta}{2} = -t\sin\frac{\theta}{2}$$

$$x_1 \sin\frac{\theta}{2} + x_2 \cos\frac{\theta}{2} = -\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} + t\cos\frac{\theta}{2} - \cos\frac{\theta}{2} + \cos\frac{\theta}{2}\sin^2\frac{\theta}{2}$$

$$= (t-1)\cos\frac{\theta}{2}$$

$$g^j_m(x_1,x_2;\theta) = \sum_{m''} \frac{\left(-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{j+m''}\left(t-\cos^2\frac{\theta}{2}\right)^{j-m''}}{\sqrt{(j+m'')!\,(j-m'')!}}\, d^j_{mm''}(\theta)$$

$$\left.\frac{\partial g^j_m}{\partial t^{j-m'}}\right|_{t=\cos^2\frac{\theta}{2}} = (-1)^{j+m'}\sqrt{\frac{(j-m')!}{(j+m')!}}\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{j+m'} d^j_{mm'}(\theta)$$

D'autre part $\quad g^j_m = (-1)^{j+m}\dfrac{\left(\sin\frac{\theta}{2}\right)^{j+m}\left(\cos\frac{\theta}{2}\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\, t^{j+m}(t-1)^{j-m}$

$$= \frac{(-1)^{2j}}{2^{2j}}\frac{\left(\sin\frac{\theta}{2}\right)^{j+m}\left(\cos\frac{\theta}{2}\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}(1+x)^{j+m}(1-x)^{j-m}$$

(on pose $x = 2t-1$)

$$\frac{\partial g^j_m}{\partial t^{j-m'}} = 2^{j-m'}\frac{\partial g_m^j}{\partial x^{j-m'}} = \frac{1}{2^{j+m'}}(-1)^{2j}\frac{\left(\sin\frac{\theta}{2}\right)^{j+m}\left(\cos\frac{\theta}{2}\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\frac{d^{j-m'}}{dx^{j-m'}}\left\{(1+x)^{j+m}(1-x)^{j-m}\right\}$$

On pose $t = \cos^2\frac{\theta}{2}$, i.e $x = \cos\theta$ $\qquad \begin{vmatrix} 1-\cos\theta = 2\sin^2\frac{\theta}{2} \\ 1+\cos\theta = 2\cos^2\frac{\theta}{2}\end{vmatrix}$

$$P_{j-m'}^{m'-m,\,m'+m}(\cos\theta) = \frac{(-1)^{j-m'}}{2^{j-m'}(j-m')!}\left(2\sin^2\frac{\theta}{2}\right)^{m-m'}\left(2\cos^2\frac{\theta}{2}\right)^{-m-m'}\left.\frac{d^{j-m'}}{dx^{j-m'}}\left\{(1+x)^{j+m}(1-x)^{j-m}\right\}\right|_{x=\cos\theta}$$

$$= \frac{(-1)^{j-m'}}{2^{j+m'}(j-m')!}\left(\sin\frac{\theta}{2}\right)^{2m-2m'}\left(\cos\frac{\theta}{2}\right)^{-2m-2m'}\left.\frac{d^{j-m'}}{dx^{j-m'}}\left\{(1+x)^{j+m}(1-x)^{j-m}\right\}\right|_{x=\cos\theta}$$

Donc $\quad \boxed{d^j_{mm'}(\theta) = \sqrt{\frac{(j+m')!\,(j-m')!}{(j+m)!\,(j-m)!}}\left(\sin\frac{\theta}{2}\right)^{m'-m}\left(\cos\frac{\theta}{2}\right)^{m'+m} P_{j-m'}^{m'-m,\,m'+m}(\cos\theta)}$