The spectral distance on Moyal Plane

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NONCOMMUTATIVE GEOMETRY

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   - Noncommutative Geometry - General features
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2. MOYAL - BASICS
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4. Noncommutative Torus - preliminaries

Noncommutative Geometry - General features

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Noncommutative Geometry - General features

- Noncommutative geometry: attempt to generalize concepts from ordinary geometry to some classes of algebras that are "sufficiently regular" to be reasonably called "noncommutative manifolds".
- Natural "noncommutative analog of space" within C*-algebras:
  Gel'fand-Naimark Theorem for commutative C* algebras.
  Noncommutative C* algebras $\longrightarrow$ Noncommutative spaces.
- Other notions have natural noncommutative analogs: e.g
  Vector bundles $\longrightarrow$ Projective modules, vector fields, connexion and curvature can be generalized, de Rham cohomology $\longrightarrow$ Cyclic cohomology,...

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- 2 ways to get Riemann distance:$(\gamma : [0,1] \to M,\ \gamma(0) = \omega_1,\ \gamma(1) = \omega_2)$

$$d_g(\omega_1, \omega_2) := \inf_{\gamma} \big(l(\gamma)\big) = \sup_{a \in C(M)} \big(|a(\omega_1) - a(\omega_2)|,\ ||[D, a]|| \leq 1\big)$$

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- LHS: Trajectories; RHS: operatorial. D, selfadjoint acts on Hilbert space of L^2 spinors, functions a act as left multiplication operator on Hilbert sp., $||[D, a]|| = ||\nabla a||_\infty$. $a \in C(M)$: coordinate algebra.

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- Commutative case: points are pure states $(a(\omega) = \omega(a))$. Re-express : $d_g(\omega_1, \omega_2) = \sup(|\omega_1(a) - \omega_2(a)|,\ ||[D, a]|| \leq 1))$

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- $(C(M), L^2(M), D)$ "encodes" (compact) Riemann geom. $\longrightarrow (\mathbb{A}, \mathcal{H}, D)$
- Tasks of NCG:
  - Describe metric differential geometry with operator language
  - (Re)construct ordinary geometry with operator language
  - Construct new geometries with NC coordinate algebras

Spectral triple

Definition 1

Spectral triple is $(\mathbb{A}, \mathcal{H}, D)$ with:
i) $\mathbb{A}$, associative involutive algebra, represented faithfully $\pi : \mathbb{A} \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ (separable) Hilbert
ii) D selfadjoint not necessarely bounded, defined on $\mathrm{Dom}(D)$ dense in $\mathcal{H}$
iii) $\pi(a)(D-\lambda)^{-1} \in \mathcal{K}(\mathcal{H}),\ \forall \lambda \notin \mathrm{Sp}(D)$
iv) $[D, \pi(a)] \in \mathcal{B}(\mathcal{H})$

-
- Often supplemented by additional conditions:
  - Regularity conditions (local index formules,...)
  - Reality conditions (reconstruction theor.,...)
- What is needed to actually compute the distance is $(\mathbb{A}, \mathcal{H}, D)$.

Metric aspect of noncommutative geometry

- We set

Definition 2 (Connes, 1994)

A spectral triple $(\mathbb{A}, \mathcal{H}, D)$ induces a distance on the space of states $\mathcal{S}(\mathbb{A})$ defined by

$$d(\omega_1, \omega_2) = \sup_{a \in \mathbb{A}} \left(|\omega_1(a) - \omega_2(a)|, \ ||[D, \pi(a)]|| \leq 1 \right) \tag{1}$$

for any $\omega_1, \omega_2 \in \mathbb{A}$.

- $(\mathcal{S}(\mathbb{A}), d)$ metric space.
- So far available studies deal only with (1-D) lattice geometry or finite geometry (matrix algebra). Very few explicit computations of distance formulas due to rapidly increasing difficulties for more sophisticated noncommutative geometries.

Past results on spectral distance

- A very simple exemple: $(\mathbb{A} = \mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, D)$. Pure states are end points of $[0,1]$. $d = \frac{1}{|a|}$ for $D =$antidiag(a,a) $(a \neq 0)$. $d = +\infty$ for $D =$diag(a,b).
- So far available studies deal only with (1-D) lattice geometry or finite geometry (matrix algebra). Very few explicit computations of distance formulas due to rapidly increasing difficulties for more sophisticated noncommutative geometries.
- First rudimentary results from physicists: 1-D lattice (Dirac operator=finite difference operator) $d \sim$lattice spacing (Dimakis, Müller-Hoissen 1993; Bimonte, Lizzi, 1994)
- Finite dimensional algebras: Exemple: $\mathbb{M}_2(\mathbb{C})$ and $(\mathbb{M}_2(\mathbb{C}), \mathbb{C}^2, D)$: d$\sim$ euclidean distance on the sphere (Iochum, Krajewski,Martinetti, 2001). Untractable for $n > 3$.
- Proposed criterium for Compact quantum metric space, Rieffel 1998-2003.
- Computation within "Moyal space" of explicit spectral distance formula provides a first exemple within a "non trivial" noncommutative geometry.
- Moyal spaces received interest from physicists.

MOYAL - BASICS

1 NONCOMMUTATIVE GEOMETRY

2 MOYAL - BASICS
- The Moyal product
- The matrix base

3 DISTANCE ON MOYAL PLANE

4 Noncommutative Torus - preliminaries

The Moyal product

- $\mathcal{S}(\mathbb{R}^2) \equiv \mathcal{S}$: Schwarz functions, $\mathcal{S}'(\mathbb{R}^2) \equiv \mathcal{S}'$, $||.||_2$, $\langle .,. \rangle$: $L^2(\mathbb{R}^2)$ norm and inner product.

Definition 3

Associative bilinear Moyal $\star$-product defined as: $\star : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, $\forall a, b \in \mathcal{S}$

$$(a \star b)(x) = \frac{1}{(\pi\theta)^2} \int d^2y d^2z \; a(x+y) b(x+t) e^{-i2y\Theta^{-1}t}$$

$$y\Theta^{-1}t \equiv y^\mu \Theta^{-1}_{\mu\nu} t^\nu, \; \Theta_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \theta \in \mathbb{R}, \; \theta \neq 0$$

Proposition 4 (see e.g Gracia-Bondia, Varilly, 1988)

One has:
i $(a \star b)^\dagger = b^\dagger \star a^\dagger$
ii) $(a, b) := \int d^2x \; (a \star b)(x) = \int d^2x \; (b \star a)(x) = \int d^2x \; a(x) b(x)$
iii) $\partial_\mu (a \star b) = \partial_\mu a \star b + a \star \partial_\mu b$.
iv) $\mathcal{A} \equiv (\mathcal{S}, \star)$ *is a non unital associative involutive Fréchet algebra.*

The matrix base

- Natural basis for $(\mathcal{S}, \star)$:

Definition 5

Matrix base: family of functions $\{f_{mn}\}_{m,n\in\mathbb{N}} \subset \mathcal{S}$ such that

$$H \star f_{mn} = \theta(m + \frac{1}{2}) f_{mn}, \;\; f_{mn} \star H = \theta(n + \frac{1}{2}) f_{mn}, \;\; H = \frac{1}{2}(x_1^2 + x_2^2), \;\; \forall m, n \in \mathbb{N}$$

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- Usefull properties (Set $\bar{z} = \frac{1}{\sqrt{2}}(x_1 - ix_2)$, $z = \frac{1}{\sqrt{2}}(x_1 + ix_2)$.)

Proposition 6

$\{f_{mn}\}_{m,n\in\mathbb{N}}$ *with* $f_{mn} = \frac{1}{(\theta^{m+n} m! n!)^{1/2}} \bar{z}^{\star m} \star f_{00} \star z^{\star n}$, $f_{00} = 2e^{-2H/\theta}$. *One has :*

$$f_{mn} \star f_{pq} = \delta_{np} f_{mq},\ f_{mn}^* = f_{nm},\ \langle f_{mn}, f_{kl} \rangle = (2\pi\theta)\delta_{mk}\delta_{nl} \tag{2}$$

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- Usefull isomorphism

Proposition 7 (Gracia-Bondia, Varilly, 1988)

Frechet algebra isomorphism between $\mathcal{A} \equiv (\mathcal{S}, \star)$ *and matrix algebra of decreasing sequences* (a_{mn}), $\forall m, n \in \mathbb{N}$ *defined by* $a = \sum_{m,n} a_{mn}f_{mn}$, $\forall a \in \mathcal{S}$, *such that the semi-norms* $\rho_k^2(a) \equiv \sum_{m,n} \theta^{2k}(m + \frac{1}{2})^k(n + \frac{1}{2})^k|a_{mn}|^2 < \infty$, $\forall k \in \mathbb{N}$.

The matrix base - II

- Within matrix base, star product is like "matrix product". For $a = \sum_{m,n} a_{mn} f_{mn}$, $b = \sum_{m,n} b_{mn} f_{mn}$, $a, b \in \mathcal{S}$, sequences $c_{mn} = \sum_p a_{mp} b_{pn}$, $\forall m, n \in \mathbb{N}$ define the function $c = \sum_{m,n} c_{mn} f_{mn} := a \star b$.

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- $\star$-product can be extended to larger space of functions. For present purpose, enough to deal with $L^2(\mathbb{R}^2)$.
- Usefull property (Set $L_a(b) := a \star b$)

Proposition 8

For any $a, b \in L^2(\mathbb{R}^2)$, $a \star b \in L^2(\mathbb{R}^2)$, $||a \star b||_2 \leq \frac{1}{2\pi\theta} ||a||_2 ||b||_2$ so that $||L_a|| \leq \frac{1}{2\pi\theta} ||a||_2$.

Proof.

Use matrix base and Cauchy-Schwartz inequality. □

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Proposition 9 (Gracia-Bondia, Varilly, 1988)

$(\mathcal{A}, \star)$ is a pre-C algebra.*

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The spectral triple

- Set: $\partial := \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2)$, $\bar{\partial} := \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2)$
- Set: $\mathcal{A} := (\mathcal{S}, \star)$, $\mathcal{H} := L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$, $D := -i\partial_\mu \otimes \sigma^\mu$,
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ D = -i\sqrt{2}\begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix}$$
- D is self-adjoint, densely defined on $\mathrm{Dom}(D) = (\mathcal{D}_{L^2} \otimes \mathbb{C}^2)$.
- Faithfull representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\pi(a) := L_a \otimes \mathbb{I}_2$.
$\pi(a)\psi = (a \star \psi_1, a \star \psi_2)$, $\forall \psi = (\psi_1, \psi_2) \in \mathcal{H}$, $\forall a \in \mathcal{A}$.
- $[D, \pi(a)] \in \mathcal{B}(\mathcal{H})$ in view of
$$[D, \pi(a)]\psi = -i\sqrt{2}\begin{pmatrix} L_{\partial a} & 0 \\ 0 & L_{\bar{\partial} a} \end{pmatrix}\begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}, \ \forall \psi = \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix} \in \mathcal{H}$$
- $\pi(a)(D - \lambda)^{-1} \in \mathcal{K}(\mathcal{H})$, $\forall a \in \mathcal{A}$, $\forall \lambda \notin \mathrm{Sp}(D)$. (Observe: $\pi(a)$ is trace class and $(D - \lambda)^{-1}$ for $\lambda \notin \mathrm{Sp}(D)$ bounded).
- Hence

Proposition 10

$(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple.

Spectral distance on the Moyal plane

- A few past studies [lattice(Dimakis, Müller-Hoissen; Bimonte, Lizzi, Sparano), finite spaces (Iochum, Krajewski, Martinetti), inspired by physics (Martinetti), quantum metric spaces (Rieffel)]

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- A very few explicit distance formulas as technically difficult to obtain when dealing with (even) not (too) trivial noncommutaitve spaces.
- The spectral distance can be defined as follows

Definition 11 (Connes 1994)

The spectral distance between any two states ω_1 and ω_2 of $\bar{\mathcal{A}}$ is defined by

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \left\{ |\omega_1(a) - \omega_2(a)|;\ ||[D, \pi(a)]||_{op} \leq 1 \right\}$$

where $||.||_{op}$ is the operator norm for the representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$.

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- Observe: Representation of $\mathcal{A}$ in the triple reducible. $\mathcal{G}_N :=\text{Span}(f_{mN})_{m\in\mathbb{N}}$, N fixed, invariant under left action of $\mathcal{A}$.

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- Observe: Representation of $\mathcal{A}$ in the triple reducible. $\mathcal{G}_N :=\text{Span}(f_{mN})_{m\in\mathbb{N}}$, N fixed, invariant under left action of $\mathcal{A}$.
- Vector state: $\omega_{mn}(a) := \frac{1}{2\pi\theta} < f_{mn}, L_a f_{mn} >= a_{mm}$. Depends only on "first index". Then, fix $N = 0$. Work with $\mathcal{G})0$

Proposition 12

The pure states of $\bar{\mathcal{A}}$ are the vector states $\omega_\psi : \bar{\mathcal{A}} \to \mathbb{C}$ defined by any unit vector $\psi \in L^2(\mathbb{R}^2)$ of the form $\psi = \sum_{m\in\mathbb{N}} \psi_m f_{m0}$, $\sum_{m\in\mathbb{N}} |\psi_m|^2 = \frac{1}{2\pi\theta}$ and one has

$$\omega_\psi(a) \equiv \big\langle (\psi, 0), \pi(a)(\psi, 0)\big\rangle = 2\pi\theta \sum_{m,n\in\mathbb{N}} \psi_m^* \psi_n a_{mn} \tag{3}$$

Proof.

i) Show that $\bar{\mathcal{A}}$ is $\star$-isomorphic (so isometrically) to $\mathcal{K}(\mathcal{G}_0)$.
ii) The result follows from Lemma [Kadison, II, p.750]: Let $\mathbb{A}$, a sub-C* of $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H}) \subseteq \mathbb{A}$, ρ a pure state of $\mathbb{A}$. Then, either $\rho = 0$ or ρ is vector state generated by some unit vector in $\mathcal{H}$. □

The unit ball

- $a \in \mathcal{A}$, $||[D, \pi(a)|| \leq 1$.
- Useful technical lemma (that extends to NC torus, Podleś sphere, $SU_q(2)$)

Lemma 13

We set $\partial a = \sum_{m,n} \alpha_{mn} f_{mn}$ *and* $\bar{\partial} a = \sum_{m,n} \beta_{mn} f_{mn}$, *for any* $a \in \mathcal{A}$ *and any unit vector* $\varphi = \sum_{m,n} \varphi_{mn} f_{mn} \in L^2(\mathbb{R}^2)$.
Assume that $||[D, \pi(a)]||_{op} \leq 1$. *Then:*
i) $|\alpha_{mn}| \leq \frac{1}{\sqrt{2}}$ *and* $|\beta_{mn}| \leq \frac{1}{\sqrt{2}}$, $\forall m, n \in \mathbb{N}$.
ii) For any radial function $a \in \mathcal{A}$ *(i.e* $a_{mn} = 0$ *if* $m \neq n$*),* $||[D, \pi(a)]||_{op} \leq 1$ *is equivalent to* $|\alpha_{mn}| \leq \frac{1}{\sqrt{2}}$ *and* $|\beta_{mn}| \leq \frac{1}{\sqrt{2}}$, $\forall m, n \in \mathbb{N}$.
iii) Let $\hat{a}(m_0) := \sum_{p,q \in \mathbb{N}} \hat{a}_{pq}(m_0) f_{pq}$, *where* $\hat{a}_{pq}(m_0) = \delta_{pq} \sqrt{\frac{\theta}{2}} \sum_{k=p}^{m_0} \frac{1}{\sqrt{k+1}}$ *with fixed* $m_0 \in \mathbb{N}$. *Let* $\mathcal{A}_+$ *denotes the set of positive elements of* $\mathcal{A}$. *Then,* $\hat{a}(m_0) \in \mathcal{A}_+$ *and* $||[D, \pi(\hat{a}(m_0))]||_{op} = 1$ *for any* $m_0 \in \mathbb{N}$.

Proof.

Go to

□

A spectral distance formula on the Moyal plane

Theorem 14

The spectral distance between any two pure states ω_m and ω_n is

$$d(\omega_m, \omega_n) = \sqrt{\frac{\theta}{2}} \sum_{k=n+1}^{m} \frac{1}{\sqrt{k}}, \ \forall m, n \in \mathbb{N}, \ n < m. \tag{4}$$

It verifies the "triangular equality"

$$d(\omega_m, \omega_n) = d(\omega_m, \omega_p) + d(\omega_p, \omega_n), \ \forall m, n, p \in \mathbb{N}. \tag{5}$$

Proof of theTheorem

Proof.

Algebraic property of matrix base yields

$$\alpha_{n+1,n} = \sqrt{\frac{n+1}{\theta}}(a_{n+1,n+1} - a_{n,n}) = \sqrt{\frac{n+1}{\theta}}(\omega_{n+1}(a) - \omega_n(a)),\ \forall n \in \mathbb{N}.$$

Use Lemma: for any a in the unit ball, $|\omega_{n+1}(a) - \omega_n(a))| \leq \sqrt{\frac{\theta}{2}}\frac{1}{\sqrt{n+1}}$, $\forall n \in \mathbb{N}$ so $d(\omega_{n+1}, \omega_n) \leq \sqrt{\frac{\theta}{2}}\frac{1}{\sqrt{n+1}}$, $\forall n \in \mathbb{N}$. This bound is saturated by any $\hat{a}(m_0)$, $m_0 \geq n$, $m_0, n \in \mathbb{N}$ defined in Lemma.Therefore: $d(\omega_{n+1}, \omega_n) = \sqrt{\frac{\theta}{2}}\frac{1}{\sqrt{n+1}}$, $\forall n \in \mathbb{N}$. Now, triangular inequality: $d(\omega_m, \omega_n) \leq \sum_{k=n+1}^{m} d(\omega_k, \omega_n)$, (assuming $n < m$). Upper bound saturated by any $\hat{a}(m_0)$, $m_0 \geq n$. Consider $|\omega_m(\hat{a}(m_0)) - \omega_n(\hat{a}(m_0)))|$, $n < m \leq m_0$.

$$|\omega_m(\hat{a}(m_0)) - \omega_n(\hat{a}(m_0)))| = \sqrt{\frac{\theta}{2}}|\sum_{k=m}^{m_0}\frac{1}{\sqrt{k+1}} - \sum_{k=n}^{m_0}\frac{1}{\sqrt{k+1}}| = \sqrt{\frac{\theta}{2}}\sum_{k=n}^{m-1}\frac{1}{\sqrt{k+1}} \tag{6}$$

Therefore, $d(\omega_m, \omega_n)$ satisfies (5). Relation (4) follows immediately. □

Some states at finite distance

- Some states at finite distance to each other

Proposition 15

Let $\mathcal{I}$ be a finite subset of $\mathbb{N}$ and let $\Lambda = \sum_{m\in\mathcal{I}\subset\mathbb{N}} \lambda_m f_{m0}$ denotes a unit vector of $L^2(\mathbb{R}^2)$. Then $d(\omega_n, \omega_\Lambda) < +\infty$, for any $n \in \mathbb{N}$.

Proof.

For any $n \in \mathbb{N}$, and any $a \in \mathcal{A}$, including any element of the unit ball, one has

$$|\omega_\Lambda(a) - \omega_n(a)| = |2\pi\theta \sum_{p,q\in\mathcal{I}} a_{pq}\lambda_p^\star\lambda_q - a_{nn}| \leq 2\pi\theta \sum_{p,q\in\mathcal{I}} |a_{pq}||\lambda_p^\star\lambda_q| + |a_{nn}|$$
$$\leq \sum_{p,q\in\mathcal{I}} |a_{pq}| + |a_{nn}|$$

(last inequality from: $|\lambda_n| \leq \frac{1}{\sqrt{2\pi\theta}}$, $\forall n \in \mathcal{I}$). Simple algebraic property of matrix base: a_{mn}'s expressible as finite sums of α_{mn} and β_{mn}. Unit ball: $|\alpha_{mn}| \leq \frac{1}{\sqrt{2\pi\theta}}$ and $|\beta_{mn}| \leq \frac{1}{\sqrt{2\pi\theta}}$. Therefore RHS bounded. □

States at infinite distance

- There are pure states at infinite distance

Definition 16

Let $\psi(s)$ family of unit vectors of $L^2(\mathbb{R}^2)$ defined by $\psi(s) := \frac{1}{\sqrt{2\pi\theta}} \sum_{m\in\mathbb{N}} \sqrt{\frac{1}{\zeta(s)(m+1)^s}} f_{m0}$ for any $s \in \mathbb{R}$, $s > 1$ ($\zeta(s)$ Riemann zeta function). Corresponding family of pure states denoted by $\omega_{\psi(s)}$, for any $s \in \mathbb{R}$, $s > 1$, with $\omega_{\psi(s)}$ as in Proposition 12.

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- At infinite distance from any ω_m

Proposition 17

$d(\omega_n, \omega_{\psi(s)}) = +\infty$, $\forall s \in]1, \frac{3}{2}]$, $\forall n \in \mathbb{N}$.

Proof of Proposition 17

Proof.

$$B(m_0;\psi,\psi') := |\omega_{\psi'}(\hat{a}(m_0)) - \omega_\psi(\hat{a}(m_0))| \leq d(\omega_\psi,\omega'_\psi),\ \forall m_0 \in \mathbb{N}. \tag{7}$$

First pick $\psi = \frac{1}{\sqrt{2\pi\theta}} f_{00} := \psi_0$. Assume that $\psi' = \psi(s)$. Then:

$$B(m_0;\psi_0,\psi(s)) = \sqrt{\frac{\theta}{2}}\Big|\sum_{m=0}^{m_0}\sum_{k=m}^{m_0}\frac{1}{\sqrt{k+1}}\frac{1}{\zeta(s)(m+1)^s} - \sum_{k=0}^{m_0}\frac{1}{\sqrt{k+1}}\Big|. \tag{8}$$

Next: "$\sum_{k=m}^{m_0} = \sum_{k=0}^{m_0} - \sum_{k=0}^{m}$"

$$B(m_0;\psi_0,\psi(s)) = \sqrt{\frac{\theta}{2}}\Big|A_1(m_0) + \frac{1}{\zeta(s)}\sum_{m=0}^{m_0}\sum_{k=0}^{m}\frac{1}{(m+1)^s\sqrt{k+1}}\Big|. \tag{9}$$

$A_1(m_0)$ positive term. Then observe

$$\frac{1}{\zeta(s)}\sum_{m=0}^{m_0}\sum_{k=1}^{m+1}\frac{1}{(m+1)^s\sqrt{k}} \geq \frac{1}{\zeta(s)}\sum_{m=0}^{m_0}\frac{\sqrt{m+1}}{(m+1)^s}, \tag{10}$$

use $\sum_{k=1}^{m+1}\frac{1}{\sqrt{k}} \geq 2(\sqrt{m+2}-1)$. $A_2(m_0)$ bounded below by quantity divergent when m_0 goes to $+\infty$ whenever $s \leq \frac{3}{2}$. Therefore: $d(\omega_0,\omega_{\psi(s)}) = +\infty$, $\forall s \in]1,\frac{3}{2}]$. Triangular inequality $d(\omega_0,\omega_{\psi(s)}) \leq d(\omega_0,\omega_n) + d(\omega_n,\omega_{\psi(s)})$, for any $n \in \mathbb{N}$ terminates the proof. □

States at infinite distance

- Distance among the $\omega_{\psi(s)}$'s is infinite.

Proposition 18

$d(\omega_{\psi(s_1)}, \omega_{\psi(s_2)}) = +\infty$, $\forall s_1, s_2 \in]1, \frac{5}{4}[\cup]\frac{5}{4}, \frac{3}{2}]$, $s_1 \neq s_2$.

Proof.

Repeated use of mean value theorem to obtain estimates of $|\omega_{\psi'}(\hat{a}(m_0)) - \omega_{\psi}(\hat{a}(m_0))|$. □

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- Weak* Topology : Basic condition to have compact quantum metric spaces as defined by Rieffel
- Remark: Spectral Triple proposed by Gayral et al. as NC analog of non compact Riemann spin manifold built from $(\mathcal{A}, \mathcal{H}, D)$. Of course, it is not Rieffel. Notice: Spectral triple has a prefered unitalization $(\mathcal{A}_1, \mathcal{H}, D)$ which is not Rieffel despite $\mathcal{A}_1$ unital.

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- Modifications of this spectral triple reinstauring compact quantum metric space [Cagnache, D'Andrea, Martinetti, Wallet 2009].

Remark

- Equivalence classes:

Definition 19

For any states ω_1 and ω_2, denote by $\approx$ the equivalence relation $\omega_1 \approx \omega_2 \iff d(\omega_1, \omega_2) < +\infty$. $[\omega]$ denotes the equivalence class of ω.

- Several equivalence classes: $[\omega_n] = [\omega_0]$, $\forall n \in \mathbb{N}$. $[\omega_\Lambda] = [\omega_0]$. In view of Proposition 17 and Proposition 18, $[\omega_{\psi(s_1)}] \neq [\omega_0]$, $\forall s_1 \in]1, \frac{3}{2}]$, and $[\omega_{\psi(s_1)}] \neq [\omega_{\psi(s_2)}]$, $\forall s_1, s_2 \in]1, \frac{5}{4}[\cup]\frac{5}{4}, \frac{3}{2}]$, $s_1 \neq s_2$.
- Therefore, uncountable infinite family of equivalence classes.
- Existence of several distinct equivalent classes implies that there is no state that is at finite distance to all other states.

Proposition 20

For any state, there is at least another state which is at infinite distance.

- This latter property applies to pure and non pure states.

Compact quantum metric space

Rieffel observation: Let commutative compact metric space (X, ρ). Lipschitz semi norm $l(f)$ on $\mathbb{A} := C(X)$. Then, one can define on $\mathcal{S}(\mathbb{A})$ a distance $\rho_l(\omega_1, \omega_2) = \sup(|(\omega_1 - \omega_2)(f)),\ l(f) \leq 1)$ such that $\lim \rho_l(\omega_n, \omega) = 0 \iff \lim(\omega_n(f) - \omega(f)) = 0,\ \forall f \in \mathbb{A}$.

Definition 21 (Rieffel, Contemp. Math. 2004)

A Compact Quantum Metric Space (CQMS) is a order unit space $\mathbb{A}$ equiped with a seminorm l such that $l(1) = 0$ and the distance defined by

$$d(\omega_1, \omega_2) = \sup\left(|\omega_1 - \omega_2(a)|,\ /\ l(a) \leq 1\right) \tag{11}$$

induced the weak* topology on the state space of $\mathbb{A}$.

- Order unit space: linear sp. of self-adjoint operators on some $\mathcal{H}$ with unit. State notion extend to this space.
- Spectral distance: sup is reached on self-adjoint elements.
- Therefore: unital spectral triple whose spectral distance induces weak* topology on $\mathcal{S}(\mathbb{A})$: CQMS.

The space of pure states - Proof

Proof.

We work with $\bar{\mathcal{A}}$.
To show ω_ψ's are pure and that all pure states are of this kind: show that $\bar{\mathcal{A}}$ is (isometrically) isomorphic to algebra of compact operators $\mathcal{K}(\mathcal{G}_0)$, since by [Kadison, 10.4.4, II, p.750] $\mathcal{K}(\mathcal{G}_0)$ is set of vector states of $\mathcal{G}_0$, actually defined in the proposition.
i) Use GNS representation $\{\pi_m, \mathcal{H}_m\}$ induced by ω_{mn}.
Since $(a^* a)_{pq} = \sum_l \bar{a}_{lp} a_{lq}$, the left kernel N_m of ω_{mn} is the ideal generated by $\{f_{pq}\}_{p\in\mathbb{N}, q\in\mathbb{N}/\{m\}}$ so that $\mathcal{H}_m := \bar{\mathcal{A}}/N_m = \mathcal{G}_m$.. As GNS repres. faithful, $\bar{\mathcal{A}}$ is *-isomorphic and so isometrically isomorphic (any injective C* morphism is isometric) to the C^*-algebra $\pi_m(\bar{\mathcal{A}}) \subset \mathcal{B}(\mathcal{H}_m)$.
ii) Let $\mathcal{I}$, the set of finite rank operators on $\mathcal{K}(\mathcal{G}_0)$. For any f_{pq}, $\pi_m(f_{pq}) \in \mathcal{I}$. So any finite rank operator can be written as a finite sum of f_{pq}. Therefore, $\mathcal{I} \subset \pi_m(\bar{\mathcal{A}})$, hence $\bar{\mathcal{I}} := \mathcal{K}(\mathcal{G}_0) \subset \overline{\pi_m(\mathcal{A})} = \pi_m(\bar{\mathcal{A}})$.
iii) Conversely, $\pi_m(\bar{\mathcal{A}}) \subset \mathcal{K}(\mathcal{G}_0)$ (use matrix base to show L_a is Hilbert-Schmidt on $\mathcal{G}_0$, therefore compact. Therefore, one has $\bar{\mathcal{A}} = \mathcal{K}(\mathcal{G}_0)$. The result follows. □

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Technical Lemma - Proof

Proof.

If $||[D,\pi(a)]||_{op} \leq 1$, then $||\partial a||_{op} \leq \frac{1}{\sqrt{2}}$ and $||\bar{\partial} a||_{op} \leq \frac{1}{\sqrt{2}}$. Use matrix base: for any $\varphi \in \mathcal{H}_0$, $||\partial a \star \varphi||_2^2 = 2\pi\theta \sum_{m,n} |\sum_p \alpha_{mp}\varphi_{pn}|^2$. From def. of $||\partial a||_{op}$, one get $\sum_{m,n} |\sum_p \alpha_{mp}\varphi_{pn}|^2 \leq \frac{1}{4\pi\theta}$ for any $\varphi \in \mathcal{H}_0$ with $\sum_{m,n} |\varphi_{mn}|^2 = \frac{1}{2\pi\theta}$. Then

$$|\sum_p \alpha_{mp}\varphi_{pn}| \leq \frac{1}{2\sqrt{\pi\theta}}, \ \forall \varphi \in \mathcal{H}_0, \ ||\varphi||_2 = 1, \ \forall m,n \in \mathbb{N} \tag{12}$$

(same for β_{mn}). Now, $|\sum_p \alpha_{mp}\varphi_{pn}| \leq \frac{1}{2\sqrt{\pi\theta}}$ true for any $\varphi \in \mathcal{H}_0$ with $||\varphi||_2 = 1$¿ One can construct $\tilde{\varphi}$ with $||\tilde{\varphi}||_2 = ||\varphi||_2$ via $\alpha_{mp}\tilde{\varphi}_{pn} = |\alpha_{mp}||\varphi_{pn}|$. Then

$$\sum_p |\alpha_{mp}||\varphi_{pn}| \leq \frac{1}{2\sqrt{\pi\theta}}, \ \forall \varphi \in \mathcal{H}_0, \ ||\varphi||_2 = 1, \ \forall m,n \in \mathbb{N} \tag{13}$$

Notice that (13) implies (12). Similar considerations apply for the β_{mn}'s. The property i) follows.

□

Technical Lemma - Proof II

Proof.

To prove ii): observe that if a is radial, one has $\alpha_{mn} = 0$ if $m \neq n+1$ (from matrix base). Then, for any unit vector $\psi \in \mathcal{H}_0$

$$||\partial a \star \psi||_2^2 = 2\pi\theta \sum_{p,q} |\sum_r \alpha_{pr}\psi_{rq}|^2 = 2\pi\theta \sum_{p,q} |\alpha_{p,p-1}\psi_{p-1,q}|^2 \leq \pi\theta \sum_{p,q\in\mathbb{N}} |\psi_{pq}|^2 \tag{14}$$

so that $||\partial a||_{op}^2 \leq \frac{1}{2}$ showing that a is in the unit ball.
To prove that $\hat{a}(m_0) \in \mathcal{A}$ defines a positive operator of $\mathcal{B}(\mathcal{H})$ for any fixed $m_0 \in \mathbb{N}$, show: $< \psi, \pi(\hat{a}(m_0))\psi > \geq 0$, $\forall \psi \in \mathcal{H}$, for any fixed $m_0 \in \mathbb{N}$. Set $\psi = (\varphi_1, \varphi_2)$, $\varphi_i \in L^2(\mathbb{R}^2)$, $i = 1, 2$ and $\varphi_i = \sum_{m,n\in\mathbb{N}} \varphi^i_{mn} f_{mn}$. A matter of standard calculation.
Finally, notice that any positive element $a \in \mathcal{A}_+$ verifies $a^\dagger = a$ so that $(\partial a)^\dagger = \bar{\partial} a$. Therefore $||[D, \pi(a)]||_{op} = \sqrt{2}||\partial a||_{op}$. Now, standard calculation shows that the only non-vanishing coefficients $\hat{\alpha}_{pq}$ in the expansion of $\partial\hat{a}(m_0)$ satisfy $\hat{\alpha}_{p+1,p} = -\frac{1}{\sqrt{2}}$, $0 \leq p \leq m_0$, for any fixed $m_0 \in \mathbb{N}$. From the very definition of $||.||_{op}$, one infers that $||\partial\hat{a}(m_0)||_{op} = \frac{1}{\sqrt{2}}$(use for instance (14)). Therefore, one obtains $||[D, \pi(\hat{a}(m_0))]||_{op} = 1$ for any $m_0 \in \mathbb{N}$. □

Properties of $(\mathcal{S}, \star))$

Theorem 22 (see Gracia-Bondia, Varilly, 1988)

$(\mathcal{S}, \star)$ is a non unital associative involutive Fréchet algebra with faithful trace and jointly continuous product.

Proof.

- Associativity and faithfull trace standard
- Continuity of $\star$ in the product topology in $\mathcal{S}$: use estimate $||a \star b||_\infty \leq ||a||_1 ||b||_1$.
- Then: prove estimates for $x^\alpha \partial^\beta (a \star b)$, $\forall \alpha, \beta \in \mathbb{N}^2$. One get: $\star$ is continuous in $\mathcal{S}$ separately so it is jointly because $\mathcal{S}$ is Fréchet.

$\square$

Properties of $(\mathcal{S}, \star))$ - II

- $\star$-product can be extended to other subspaces of $\mathcal{S}'$ (use duality and continuity of $\star$ on $\mathcal{S}$).

- Then, for any $a \in \mathcal{G}_{s,t}$ and $b \in \mathcal{G}_{q,r}$, $b = \sum_{m,n} b_{mn} f_{mn}$, $t + q \geq 0$, the sequences $c_{mn} = \sum_p a_{mp} b_{pn}$, $\forall m, n \in \mathbb{N}$ define the functions $c = \sum_{m,n} c_{mn} f_{mn}$, $c \in \mathcal{G}_{s,r}$ [See e.g Gracia-Bondia, Varilly, JMP 1988].

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Properties of $(\mathcal{S}, \star))$ - II

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- Convenient: Hilbert spaces $\mathcal{S} \subset \mathcal{G}_{s,t} \subset \mathcal{S}'$, $s, t \in \mathbb{R}$,
  $\mathcal{G}_{s,t} = \{a = \sum a_{mn} f_{mn} \in \mathcal{S}' \ / \ ||a||^2_{s,t} = \sum_{m,n} \theta^{s+t} (m+\frac{1}{2})^s (n+\frac{1}{2})^t |a_{mn}|^2 < \infty\}$

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- Uses: $||a \star b||_{s,r} \leq ||a||_{s,t}||b||_{q,r}$, $t + q \geq 0$ and $||a||_{u,v} \leq ||a||_{s,t}$ if $u \leq s$, $v \leq t$.
- Then, for any $a \in \mathcal{G}_{s,t}$ and $b \in \mathcal{G}_{q,r}$, $b = \sum_{m,n} b_{mn} f_{mn}$, $t + q \geq 0$, the sequences $c_{mn} = \sum_p a_{mp} b_{pn}$, $\forall m, n \in \mathbb{N}$ define the functions $c = \sum_{m,n} c_{mn} f_{mn}$, $c \in \mathcal{G}_{s,r}$ [See e.g Gracia-Bondia, Varilly, JMP 1988].

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Noncommutative Torus - preliminaries

1 NONCOMMUTATIVE GEOMETRY

2 MOYAL - BASICS

3 DISTANCE ON MOYAL PLANE

4 **Noncommutative Torus - preliminaries**
- basic properties
- Pure states on noncommutative torus
- Preliminary results - Spectral distance on NC Torus

The noncommutative torus

Definition 23 (For reviews see, e.g Landi, Gracia-Bondia, Varilly)

$\mathfrak{A}_\theta^2$ universal C*-algebra generated by u_1, u_2 with $u_1 u_2 = e^{i2\pi\theta} u_2 u_1$. Algebra of the noncommutative torus $\mathbb{T}_\theta^2$ is the dense (unital) pre-C* subalgebra of $\mathfrak{A}_\theta^2$ defined by $\mathbb{T}_\theta^2 = \{a = \sum_{i,j\in\mathbb{Z}} a_{ij} u_1^i u_2^j \ / \ \sup_{i,j\in\mathbb{Z}}(1+i^2+j^2)^k |a_{ij}|^2 < \infty\}$.

- Weyl generators defined by $U^M \equiv e^{-i\pi m_1 \theta m_2} u_1^{m_1} u_2^{m_2}$, $\forall M = (m_1, m_2) \in \mathbb{Z}^2$. For any $a \in \mathbb{T}_\theta^2$, $a = \sum_{m\in\mathbb{Z}^2} a_M U^M$. Let δ_1 and δ_2: canonical derivations $\delta_a(u_b) = i2\pi u_a \delta_{ab}$, $\forall a, b \in \{1,2\}$. One has $\delta_b(a^*) = (\delta_b(a))^*$, $\forall b = 1, 2$.

Proposition 24

One has for any $M, N \in \mathbb{Z}^2$, $(U^M)^* = U^{-M}$, $U^M U^N = \sigma(M,N) U^{M+N}$ where the commutation factor $\sigma : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ satisfies

$$\sigma(M+N, P) = \sigma(M,P)\sigma(N,P), \ \sigma(M, N+P) = \sigma(M,N)\sigma(M,P), \ \forall M, N, P \in \mathbb{Z}^2$$

$$\sigma(M, \pm M) = 1, \ \forall M \in \mathbb{Z}^2$$

$$\delta_a(U^M) = i2\pi m_a U^M, \ \forall a = 1, 2, \ \forall M \in \mathbb{Z}^2$$

The noncommutative torus

- Let τ be tracial state:
  For any $a = \sum_{M \in \mathbb{Z}^2} a_M U^M \in \mathbb{T}^2_\theta$, $\tau : \mathbb{T}^2_\theta \to \mathbb{C}$, $\tau(a) = a_{0,0}$.

The noncommutative torus

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- $\mathcal{H}_\tau$: GNS Hilbert space (completion of $\mathbb{T}^2_\theta$ in the Hilbert norm induced by $< a, b > \equiv \tau(a^* b)$). One has $\tau(\delta_b(a)) = 0$, $\forall b = 1, 2$.

The noncommutative torus

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- The even real spectral triple:
  $$(\mathbb{T}^2_\theta, \mathcal{H}, D; J, \Gamma)$$
  $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^2$. One has $\delta_b^\dagger = -\delta_b$, $\forall b = 1, 2$, in view of $< \delta_b(a), c >= \tau((\delta_b(a)^* c) = \tau(\delta_b(a^*)c) = -\tau(a^*\delta_b(c)) = - < a, \delta_b(c) >$ for any $b = 1, 2$ and $\delta_b(a^*) = (\delta_b(a))^*$.

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- Define $\delta = \delta_1 + i\delta_2$ and $\bar{\delta} = \delta_1 - i\delta_2$. D: unbounded self-adjoint Dirac operator $D = -i \sum_{b=1}^2 \delta_b \otimes \sigma^b$, densely defined on $\mathrm{Dom}(D) = (\mathbb{T}^2_\theta \otimes \mathbb{C}^2) \subset \mathcal{H}$.
  $$D = -i \begin{pmatrix} 0 & \delta \\ \bar{\delta} & 0 \end{pmatrix}$$

The noncommutative torus

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- Define $\delta = \delta_1 + i\delta_2$ and $\bar{\delta} = \delta_1 - i\delta_2$. D: unbounded self-adjoint Dirac operator $D = -i\sum_{b=1}^2 \delta_b \otimes \sigma^b$, densely defined on $\mathrm{Dom}(D) = (\mathbb{T}^2_\theta \otimes \mathbb{C}^2) \subset \mathcal{H}$.
  $$D = -i\begin{pmatrix} 0 & \delta \\ \bar{\delta} & 0 \end{pmatrix}$$
- Faithfull representation $\pi : \mathbb{T}^2_\theta \to \mathcal{B}(\mathcal{H}) : \pi(a) = L(a) \otimes \mathbb{I}_2$, $\pi(a)\psi = (a\psi_1, a\psi_2)$, $\psi = (\psi_1, \psi_2) \in \mathcal{H}$, $\forall a \in \mathbb{T}^2_\theta$. $L(a)$: left multiplication operator by any $a \in \mathbb{T}^2_\theta$. $\pi(a)$ and $[D, \pi(a)]$ bounded on $\mathcal{H}$ for any $\mathbb{T}^2_\theta$.
  $$[D, \pi(a)]\psi = -i(L(\delta_b(a)) \otimes \sigma^b)\psi = -i\begin{pmatrix} L(\delta(a)) & 0 \\ 0 & L(\bar{\delta}(a)) \end{pmatrix}\begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} \quad (15)$$

Pure states on noncommutative torus

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Pure states on noncommutative torus

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- Consider rational case: $\theta = \frac{p}{q}$, $p < q$, p and q relatively prime, $q \neq 1$. Set $\mathbb{T}^2_{\frac{p}{q}} \equiv T_{p/q}$ [see e.g Connes, Landi, Rieffel]. Unitary equivalence classes of irreps. $T_{p/q}$ classified by a torus parametrized by (α, β). Irreps. given by $\pi_{\alpha,\beta} : T_{p/q} \to \mathbb{C}^q$, $\alpha, \beta \in \mathbb{C}$ unitaries and $\pi_{\alpha,\beta}(u_1), \pi_{\alpha,\beta}(u_2) \in \mathbb{M}_q(\mathbb{C})$ are the usual clock and shift matrices in the basis defined by $\{e_k = \beta^{-k/q} u_2^k e_0\}$, $\forall k \in \{0, 1, ..., q-1\}$ and $u_1 e_0 = \alpha^{1/q} e_0$.

Proposition 25

The set of pure states of the rational noncommutative torus is exactly the set of vector states $\omega^{\psi}_{\alpha,\beta} : T_{p/q} \to \mathbb{C}$

$$\omega^{\psi}_{\alpha,\beta}(a) = (\psi, \pi_{\alpha,\beta}(a)\psi), \ \forall \psi \in \mathbb{C}^q, \ ||\psi|| = 1 \tag{16}$$

where ψ *is given up to an overall phase. The pure states are then classified by a bundle over a commutative torus parametrized by* (α, β) *with fiber* $P(\mathbb{C}^q)$.

Proof.

By standard results on C*-algebras, any irrep. $(\pi_{\alpha,\beta}, \mathbb{C}^q)$ is unitarily equivalent to the GNS representation $(\omega_\psi, \pi_{\alpha,\beta})$ for any $\psi \in \mathbb{C}^q$. Then, the ω_ψ are pure states. Write now $\omega^{\psi}_{\alpha,\beta}(a) = (\psi, \pi_{\alpha,\beta}(a)\psi)$ for any $a \in T_{p/q}$. □

Preliminary results - Spectral distance on NC Torus

- One has

Lemma 26

Set $\delta(a) = \sum_{N\in\mathbb{Z}^2} \alpha_N U^N$. *One has* $\alpha_N = i2\pi(n_1 + in_2)a_N,\ \forall N = (n_1, n_2) \in \mathbb{Z}^2$.
i) For any a in the unit ball, $||[D, \pi(a)||_{op} \leq 1$ *implies* $|\alpha_N| \leq 1$, $\forall N \in \mathbb{Z}^2$. *Similar results hold for* $\bar{\delta}(a)$.
ii) The elements $\hat{a}^M \equiv \frac{U^M}{2\pi(m_1+im_2)}$ *verify* $||[D, \pi(\hat{a}^M)||_{op} = 1$, $\forall M = (m_1, m_2) \in \mathbb{Z}^2$, $M \neq (0, 0)$

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- Indeed

Proof.

The relation involving α_N obvious. Then, $||[D, \pi(a)||_{op} \leq 1$ is equivalent to $||\delta(a)||_{op} \leq 1$ and $||\bar{\delta}(a)||_{op} \leq 1$ in view of (15). For any $a \in \mathfrak{A}_\theta^2$ and any unit $\psi = \sum_{N\in\mathbb{Z}^2} \psi_N U^N \in \mathcal{H}_\tau$, one has $||\delta(a)\psi||^2 = \sum_{N\in\mathbb{Z}^2} |\sum_{P\in\mathbb{Z}^2} \alpha_P \psi_{N-P}\sigma(P, N)|^2$. Then $||\delta(a)||_{op} \leq 1$ implies $|\sum_{P\in\mathbb{Z}^2} \alpha_P \psi_{N-P}\sigma(P, N)| \leq 1$, for any $N \in \mathbb{Z}^2$ and any unit $\psi \in \mathcal{H}_\tau$. By a straighforward adaptation of the proof carried out for ii) of Lemma **??**, this implies $|\alpha_M| \leq 1$, $\forall M \in \mathbb{Z}^2$. This proves ii). Finally, iii) stems simply from an elementary calculation. □

Preliminary results - Spectral distance on NC Torus

- The following proerty holds

Proposition 27

Let the familly of unit vectors $\Phi_M = (\frac{1+U^M}{\sqrt{2}}, 0) \in \mathcal{H}$, $\forall M \in \mathbb{Z}^2$, $M \neq (0,0)$ *generating the family of vector states of* $\mathbb{T}^2_\theta$

$$\omega_{\Phi_M} : \mathbb{T}^2_\theta \to \mathbb{C},\ \omega_{\Phi_M}(a) \equiv (\Phi_M, \pi(a)\Phi_M)_{\mathcal{H}} = \frac{1}{2} < (1+U^M), (a + aU^M) > \quad (17)$$

The spectral distance between any state ω_{Φ_M} *and the tracial state is*

$$d(\omega_{\Phi_M}, \tau) = \frac{1}{2\pi|m_1 + im_2|},\ \forall M = (m_1, m_2) \in \mathbb{Z}^2,\ M \neq (0,0) \quad (18)$$

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- Sketch

Proof.

Set $a = \sum_{N \in \mathbb{Z}^2} a_N U^N$. Using Proposition 24 yields $\omega_{\Phi_M}(a) = \tau(a) + \frac{1}{2}(a_M + a_{-M})$. This, combined with Lemma 26 yields $d(\omega_{\Phi_M}, \tau) \leq \frac{1}{2\pi|m_1+im_2|}$. Upper bound obviously saturated by the element $\hat{a}^M$ of iii) of Lemma 26 which belongs to the unit ball. □