Recents results in gauge theories on noncommutative Moyal spaces

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works in coll. with **A. de Goursac, T. Masson, A. Tanasa, R. Wulkenhaar**

arXiv:hep-th/0703075, 0708.2471[hep-th], 0709.3950[hep-th]

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Overview

- Attempt to construct possible candidate(s) for renormalisable actions for gauge theories on NC $D=4$ Moyal "space". The NC analog of the Yang-Mills action $\int d^4x(F_{\mu\nu}\star F_{\mu\nu})(x)$ has UV/IR mixing which spoilts renormalisability.

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  $$S_f\sim\int d^4x\Big(\frac{\alpha}{4g^2}F_{\mu\nu}\star F_{\mu\nu}+\frac{\Omega'}{4g^2}\{\mathcal{A}_\mu,\mathcal{A}_\nu\}^2_\star+\frac{\kappa}{2}\mathcal{A}_\mu\star\mathcal{A}_\mu\Big)$$
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- 2nd part: Study vacuum configurations. Much simpler to start in D=2 scalar models with harmonic terms (A.de Goursac, A.Tanasa, JCW, [hep-th/0709.3950], Eur.Phys.J.C to appear). Recent results for vacuum configurations for above D=2 gauge theories (A.de Goursac, JCW, R.Wulkenhaar, work in progress).

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- 3rd part: Attempt to clarify the role(s) of $\mathcal{A}_\mu$. Modification of the "derivation-based" differential calculus on the Moyal algebras leads to "Yang-Mills-Higgs" type models (E.Cagnache, T.Masson, JCW, hep-th to appear).

Content

The noncommutative algebraic set-up

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- Der($\mathcal{M}$): linear space of derivations of some $\mathcal{M}$ with associative product $\star$, that is linear maps satisfying Leibnitz rule
$$X : \mathcal{M} \to \mathcal{M}, \qquad X(a \star b) = X(a) \star b + a \star X(b), \quad \forall a, b \in \mathcal{M} \qquad (1)$$
$\exists$ Lie Bracket on Der($\mathcal{M}$) defined by $[X, Y]_D(a) \equiv X(Y(a)) - Y(X(a))$ (i).
Der($\mathcal{M}$) is a module over $\mathcal{Z}(\mathcal{M})$ ($(zX)(a) = z \star (X(a))$, $\forall z \in \mathcal{Z}(\mathcal{M})$).

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Der($\mathcal{M}$) is a module over $\mathcal{Z}(\mathcal{M})$ ($(zX)(a){=}z \star (X(a))$, $\forall z \in\mathcal{Z}(\mathcal{M})$.
- From any Lie subalgebra $\mathcal{G}{\subset}$Der($\mathcal{M}$) (also a $\mathcal{Z}(\mathcal{M})$-submodule), construction of a differential calculus can be performed. [Space of 0-forms identified with $\mathcal{M}$, action of the differential d on 0-forms and 1-forms ($\mathcal{Z}(\mathcal{M})$-linear maps from $\mathcal{G}$ to $\mathcal{M}$) defined $\forall X, Y {\in} \mathcal{G}$ by $d\omega_0(X){=}X(\omega_0)$, $d\omega_1(X, Y) {=}X(\omega_1(Y)) - Y(\omega_1(X)) - \omega_1([X, Y]_D)$(ii).
$d^2{=}0$ thanks to (i) and (ii). Can be extended to n-forms,$\mathcal{Z}(\mathcal{M})$-multilinear antisymmetric maps from $\mathcal{G}$ to $\mathcal{M}$.]

Noncommutative connections, curvatures

- Once $\mathcal{M}$ equipped with diff. calculus related to $\mathcal{G}\subset\text{Der}(\mathcal{M})$, construction of NC connections and curvatures can be done [see: Connes, Dubois-Violette, Kerner, Madore]. Choose some (projective) right-module $\mathcal{H}$ over $\mathcal{M}$.

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- NC connection: linear map $\nabla_X : \mathcal{H} \to \mathcal{H}$ verifying ($a\in\mathcal{M}$, $m\in\mathcal{H}$, $X, Y\in\mathcal{G}$):

$$\nabla_X(m \star a) = \nabla_X(m) \star a + m \star X(a) \tag{2}$$

$$\nabla_{X+Y}(m) = \nabla_X(m) + \nabla_Y(m), \nabla_{(z\star X)}(m) = z \star \nabla_X(m) \tag{3}$$

[Recall Der($\mathcal{M}$) $\mathcal{Z}(\mathbb{M})$-module; (3) reflects ∇_X is a morphism of module)]
Curvature: $F_{(X,Y)}(m) \equiv [\nabla_X, \nabla_Y](m) - \nabla_{[X,Y]_D}(m)$

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- From now on, $\mathcal{M}$ is the Moyal algebra. Recall $\mathcal{M}=\mathcal{L}\cap\mathcal{R}$; $\mathcal{L}$ (resp. $\mathcal{R}$): subspace of elements of $\mathcal{S}'(\mathbb{R}^D)$ whose multiplication from right (resp. left) by any Schwarz function is Schwartz. [see e.g Gracia-Bondia, Varilly, J.M.P 1988; Grossmann et al., Ann. Inst. Fourier 1968].

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- We now assume: $\mathcal{H}=\mathcal{M}$. Then, ∇_X determined by $\nabla_X(\mathbb{I})$, $\mathbb{I}$: the unit $\in\mathcal{M}$. Indeed, one has from (2)
$$\nabla_X(a) = \nabla_X(\mathbb{I})\star a + X(a), \quad \forall a\in\mathcal{M}, \quad \forall X\in\mathcal{G} \tag{4}$$
$\nabla_X(\mathbb{I})$ will serve as a NC analog of a gauge potential.

Gauge transformations

- Convenient hermitian structure is $h_0(a_1, a_2) = a_1^\dagger \star a_2$ so that ∇ in (2) hermitean provided $(\nabla_X(\mathbb{I}))^\dagger = -\nabla_X(\mathbb{I})$.

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- Gauge transformations are defined by the automorphisms of the "module $\mathcal{M}$" preserving the hermitian structure h: $\gamma \in Aut_h(\mathcal{M})$. One has
$$\gamma(a) = \gamma(\mathbb{I} \star a) = \gamma(\mathbb{I}) \star a \ , \quad \forall a \in \mathcal{M}$$
$$h_0\big(\gamma(a_1), \gamma(a_2)\big) = h_0(a_1, a_2) \quad \forall a_1, a_2 \in \mathcal{M}$$
This implies
$$\gamma(\mathbb{I})^\dagger \star \gamma(\mathbb{I}) = \mathbb{I}$$
so that the gauge transformations are determined by $\gamma(\mathbb{I}) \in \mathcal{U}(\mathcal{M})$, where $\mathcal{U}(\mathcal{M})$ is the group of unitary elements of $\mathcal{M}$. From now on, we set $\gamma(\mathbb{I}) \equiv g$.

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- The action of $\mathcal{U}(\mathcal{M})$ on ∇_X and curvature are

$$(\nabla_X)^\gamma(a) = \gamma(\nabla_X(\gamma^{-1}(a))), \quad \forall a \in \mathcal{M}, \quad \forall X \in \mathcal{G} \qquad (6)$$

$$(F_{(X,Y)}(a))^\gamma = g \star F_{(X,Y)}(a) \star g^\dagger \qquad (7)$$

  This yields

$$(\nabla_X(\mathbb{I}))^\gamma = g \star \nabla_X(\mathbb{I}) \star g^\dagger + g \star X(g^\dagger), \quad \forall g \in \mathcal{U}(\mathcal{M}), \quad \forall X \in \mathcal{G} \qquad (8)$$

Canonical gauge-invariant connections

- Existence of inner derivations (9) implies existence of gauge invariant connections [cf. Dubois-Violette, Kerner, Madore; Dubois-Violette, Masson]. All derivations of Moyal algebra are inner, i.e for any $X \in \text{Der}(\mathcal{M})$:

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- Here, gauge-invariant connection defined by

$$\nabla_X^{inv}(\mathbb{I}) = -\eta_X, \quad \forall X \in \mathcal{G} \tag{10}$$

$$\nabla_X^{inv}(a) = \nabla_X^{inv}(\mathbb{I}) \star a + [\eta_X, a]_\star = -a \star \eta_X \tag{11}$$

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- Tensor forms $\mathcal{A}_X$ (covariant coordinates):

$$(\nabla_X - \nabla_X^{inv})(a) \equiv \mathcal{A}_X \star a = (\nabla_X(\mathbb{I}) + \eta_X) \star a \tag{12}$$

$$(\mathcal{A}_X)^\gamma = g \star \mathcal{A}_X \star g^\dagger \tag{13}$$

Curvature takes the form

$$F_{(X,Y)}(a) = ([\mathcal{A}_X, \mathcal{A}_Y]_\star - \mathcal{A}_{[X,Y]_D} - ([\eta_X, \eta_Y]_\star - \eta_{[X,Y]_D})) \star a \tag{14}$$

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- The tensor form ("covariant coordinates") and curvature are

$$\mathcal{A}_\mu = -i(A_\mu - \xi_\mu) \equiv -i\mathcal{A}^0_\mu \tag{15}$$

$$F_{\mu\nu} = -i\Theta^{-1}_{\mu\nu} + [\mathcal{A}_\mu, \mathcal{A}_\nu]_\star = -i(\Theta^{-1}_{\mu\nu} - i[\mathcal{A}^0_\mu, \mathcal{A}^0_\nu]_\star) \equiv -iF^0_{\mu\nu}$$

$$F^0_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$$

The gauge transformations are given by

$$(\mathcal{A}^0_\mu)^g = g \star \mathcal{A}^0_\mu \star g^\dagger, \quad (F^0_{\mu\nu})^g = g \star F^0_{\mu\nu} \star g^\dagger$$

Noncommutative Induced gauge theories

Motivations

- Start from the complex-valued φ_4^4 with harmonic term.
[Grosse, Wulkenhaar; Gurau, Magnen, Rivasseau, Vignes-Tourneret]: $(\widetilde{x}_\mu = 2\Theta^{-1}_{\mu\nu}x_\nu)$

$$S(\phi) = \int d^4x \big(\partial_\mu\phi^\dagger \star \partial_\mu\phi + \Omega^2(\widetilde{x}_\mu\phi)^\dagger \star (\widetilde{x}_\mu\phi) + m^2\phi^\dagger \star \phi\big)(x) + S_{int}$$

- Couple $S(\phi)$ to external gauge potential A_μ via minimal coupling prescription (de Goursac, JCW, Wulkenhaar): $\partial_\mu\phi \mapsto \nabla^A_\mu\phi = \partial_\mu\phi - iA_\mu \star \phi$,
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- Next step: Compute at the one-loop order the effective action $\Gamma(A)$ obtained by integrating over the scalar field ϕ in $S(\phi, A)$, for any value of $\Omega \in [0,1]$
- Goals:
  - Guess possible form(s) for a candidate as a renormalisable gauge action
  - Is there some additional terms that appear in the action, beyond the expected $F_{\mu\nu} \star F_{\mu\nu}$.
  - How does the harmonic term survive in the resulting effective action?

The one-loop effective action

- The effective action is formally obtained through the evaluation of the following functional integral
$$e^{-\Gamma(A)} \equiv \int D\phi D\phi^\dagger e^{-S(\phi,A)} = \int D\phi D\phi^\dagger e^{-S(\phi)} e^{-S_{int}(\phi,A)},$$
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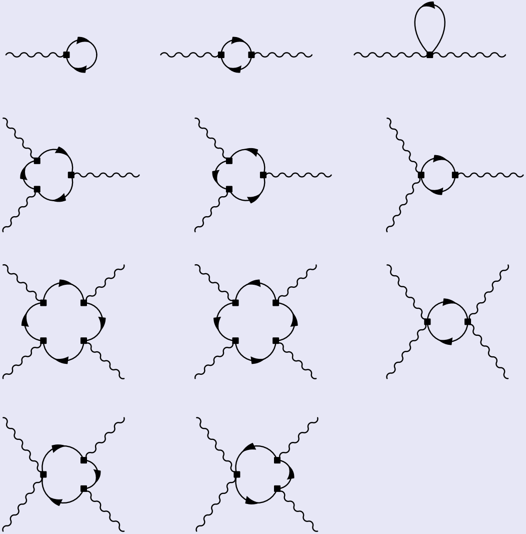
- The effective action $\Gamma_{1loop}(A)$ can be conveniently obtained in the x-space formalism. Compute relevant diagrams using the Mehler-type propagator $C(x,y) \equiv \langle \phi(x)\phi^\dagger(y)\rangle$ (set $\widetilde{\Omega} \equiv 2\frac{\Omega}{\theta}$ and $x \wedge y \equiv 2x_\mu \Theta^{-1}_{\mu\nu} y_\nu$)
$$C(x,y) = \frac{\Omega^2}{\pi^2\theta^2} \int_0^\infty \frac{dt}{\sinh^2(2\widetilde{\Omega}t)} \exp^{(-\frac{\widetilde{\Omega}}{4}\coth(\widetilde{\Omega}t)(x-y)^2 - \frac{\widetilde{\Omega}}{4}\tanh(\widetilde{\Omega}t)(x+y)^2 - m^2 t)}$$
combined with the vertex whose generic expression is
$$\int d^4x (f_1 \star f_2 \star f_3 \star f_4)(x) = \frac{1}{\pi^4\theta^4} \int \prod_{i=1}^4 d^4x_i f_1(x_1) f_2(x_2) f_3(x_3) f_4(x_4)$$
$$\times\, \delta(x_1 - x_2 + x_3 - x_4) e^{-i\sum_{i<j}(-1)^{i+j+1} x_i \wedge x_j}.$$

Diagramatics



The structure of the effective action

- The result for any $\Omega\in[0,1]$ can be writen as

$$\Gamma(A) = \frac{\Omega^2}{4\pi^2(1+\Omega^2)^3}\left(\int d^4u\ (\mathcal{A}_\mu\star\mathcal{A}_\mu-\frac{1}{4}\widetilde{u}^2)\right)\left(\frac{1}{\epsilon}+m^2\ln(\epsilon)\right)$$
$$-\frac{(1-\Omega^2)^4}{192\pi^2(1+\Omega^2)^4}\left(\int d^4u\ F_{\mu\nu}\star F_{\mu\nu}\right)\ln(\epsilon)$$
$$+\frac{\Omega^4}{8\pi^2(1+\Omega^2)^4}\left(\int d^4u\ (F_{\mu\nu}\star F_{\mu\nu}+\{\mathcal{A}_\mu,\mathcal{A}_\nu\}_\star^2-\frac{1}{4}(\widetilde{u}^2)^2)\right)\ln(\epsilon)+\dots,$$

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$$-\frac{(1-\Omega^2)^4}{192\pi^2(1+\Omega^2)^4}\left(\int d^4u\ F_{\mu\nu}\star F_{\mu\nu}\right)\ln(\epsilon)$$
$$+\frac{\Omega^4}{8\pi^2(1+\Omega^2)^4}\left(\int d^4u\ (F_{\mu\nu}\star F_{\mu\nu}+\{\mathcal{A}_\mu,\mathcal{A}_\nu\}_\star^2-\frac{1}{4}(\widetilde{u}^2)^2)\right)\ln(\epsilon)+\dots,$$

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- It involves a mass-type term for the gauge potential A_μ

The structure of the effective action II

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- Appears to be related possibly to a spectral triple (Grosse, Wulkenhaar).
- Next problem that must be solved: Vacuum determination. Appears to be (at least technically) difficult.

Vacuum configurations

The harmonic ϕ^4-model

- D=2 action for the harmonic ($\mathbb{R}$-valued) ϕ^4-model (λ>0) and eqn of motion

$$S(\phi) = \int d^2x \frac{1}{2}\partial_\mu\phi\star\partial_\mu\phi + \frac{\Omega^2}{2}(\widetilde{x}_\mu\phi)\star(\widetilde{x}_\mu\phi) - \frac{\mu^2}{2}\phi\star\phi + \lambda\phi\star\phi\star\phi\star\phi \quad (16)$$

$$-\partial^2\phi + \Omega^2\widetilde{x}^2\phi - \mu^2\phi + 4\lambda\phi\star\phi\star\phi = 0 \quad (17)$$

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$f_{mn}(r,\varphi)=(-1)^m 2\sqrt{\frac{m!}{n!}}e^{i\varphi(n-m)}(\sqrt{\frac{2}{\theta}}r)^{(n-m)}L_m^{n-m}(\frac{2r^2}{\theta})e^{-r^2/\theta}$, $f^\dagger_{mn}=f_{nm}$,
$f_{mn}\star f_{pq}=\delta_{np}f_{mq}$, $\int d^2xf_{mn}=2\pi\theta\delta_{mn}$ - Set $\partial=\frac{1}{\sqrt{2}}(\partial_1-i\partial_2)$, $\bar{\partial}=\frac{1}{\sqrt{2}}(\partial_1+i\partial_2)$,
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- Eqn of motion in the matrix basis $\phi(x)=\sum_{m,n\in\mathbb{N}}\phi_{mn}f_{mn}(x)$
$$\frac{4}{\theta}(m+n+1)\phi_{mn} - \mu^2\phi_{mn} + 4\lambda\phi_{mk}\phi_{kl}\phi_{ln} = 0 \quad (19)$$

Vacuum configurations

- Look for radial solutions $v(x)=\sum_{m\in\mathbb{N}} a_m f_{mm}(x)$. Eqn. of motion yields

$$a_m\big(a_m^2+\frac{1}{\lambda\theta}(2m+1-\frac{\mu^2}{\mu_0^2})\big)=0,\quad \mu_0^2=\frac{4}{\theta},\quad m\in\mathbb{N} \tag{20}$$

so that $a_m=0$ or $a_m^2=\frac{1}{\lambda\theta}(\frac{\mu^2}{\mu_0^2}-2m-1)$. Consistency requires RHS≥ 0. This yields $\frac{1}{2}(\frac{\mu^2}{\mu_0^2}-1)\geq m$ ($m\in\mathbb{N}$!) so that the sum is truncated: $v(x)=\sum_{m=0}^{M} a_m f_{mm}(x)$ with $M\equiv[[\frac{1}{2}(\frac{\mu^2}{\mu_0^2}-1)]]$.

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- Expanding the action around $v(x)$, one has $v(x)$ a minimum of the action provided the resulting quadratic part S_q is positive.

$$S_q=\sum_{m,n,p,q\in\mathbb{N}}\phi_{mn}\Gamma_{mn,pq}\phi_{pq},\quad \Gamma_{mn,pq}=\Gamma_{mn}\delta_{mp}\delta_{nq} \tag{21a}$$

$$\Gamma_{mn}=\sum_{m,n\in\mathbb{N}}4\pi\big(m+n+1-\frac{\mu^2}{\mu_0^2}+\lambda\theta\sum_{p=0}^{M}a_p^2(\delta_{mp}+\delta_{np})+\lambda\theta\sum_{p,q=0}^{M}a_pa_q\delta_{mp}\delta_{nq}\big) \tag{21b}$$

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  1)$M<0$. Whenever $\mu^2<\mu_0^2$. $a_m=0$, $\forall m$ and $\Gamma_{mn}=4\pi(m+n+1-\frac{\mu^2}{\mu_0^2})>0$.

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- Summary:
  Whenever $\mu^2<\mu_0^2$, $v=0$ is the (global) minimum while in the commutative situation (or when $\Omega=0$ ie, no harmonic term), vacuum configurations $v \neq 0$ (that trigger SSB) are supported. In some sense, the presence of a harmonic term prevents SSB to occur.

  Whenever $\mu^2>\mu_0^2$, the action has a non trivial vacuum configuration given by

$$v(x) = \sum_{m=0}^{M} a_m f_{mm}(x), \quad a_m^2 = \frac{1}{\lambda\theta}(\frac{\mu^2}{\mu_0^2} - 2m - 1) \tag{22}$$

Yang-Mills-Higgs type models on Moyal spaces

Basic observation

- $\mathcal{G}_0$: $[\partial_\mu, \partial_\nu]_D=0$ leads to the simplest diff. calculus on $\mathcal{M}$.
  ($[\partial_\mu, \partial_\nu]_D(a)=0=[[\xi_\mu, \xi_\nu]_\star, a]_\star$ trivially verified. $\eta_X \rightarrow$"η_{∂_μ}"$=\eta_\mu=\xi_\mu$).

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- Observe $\mathcal{G}_0$ is linked with $[x_\mu, f]_\star = i\Theta_{\mu\nu}\partial^\nu f$ which can be interpreted as Lie derivative along $(V_\mu)_\nu$ such that $\partial^\nu(V_\mu)_\nu=0$, i.e Hamiltonian vector field linked with area-preserving diffeomorphisms. A.P.D. can also be generated from polynomials of degree 2: $[(x_\mu.x_\nu), a]_\star = i(x_\mu\Theta_{\nu\beta}+x_\nu\Theta_{\mu\beta})\partial_\beta a \equiv L_W(a)$ where $(W_{(\mu\nu)})_\beta$ verifies $\partial^\beta(W_{(\mu\nu)})_\beta=0$. This would be no longer true for degree $\geq$3. Not too surprising because the Moyal bracket $[a, b]_\star$ reduces to the Poisson bracket $\{a, b\}_{PB} = \Theta^{\mu\nu}\frac{\partial a}{\partial_\mu}\frac{\partial b}{\partial_\nu}$ when restricted to polynomials of degree 2.

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- Suggest to consider the differential calculus generated by those polynomials with degree 2: $[(x_\mu . x_\nu), a]_\star$ combined with $X(a) = [\eta_X, a]_\star$ yields a new diff. calculus.

Symplectic algebra of derivations

- Case D=2 to simplify the presentation. Algebra of derivations generated by
$$\eta X_1 = \frac{i}{4\sqrt{2}\theta}(x_1^2 + x_2^2), \quad \eta X_2 = \frac{i}{4\sqrt{2}\theta}(x_1^2 - x_2^2), \quad \eta X_3 = \frac{i}{2\sqrt{2}\theta}(x_1 x_2) \quad (23)$$
and satisfying the commutation rules for a symplectic algebra $sp(2, \mathbb{R})$.
Extension to any D straighforward and yields of course $sp(D, \mathbb{R})$.
$$[\eta X_1, \eta X_2]_\star = \frac{1}{\sqrt{2}}\eta X_3, \quad [\eta X_2, \eta X_3]_\star = -\frac{1}{\sqrt{2}}\eta X_1, \quad [\eta X_3, \eta X_1]_\star = \frac{1}{\sqrt{2}}\eta X_2 \quad (24)$$

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- Enlarge with inhomogeneous "spatial part" with those ∂_μ to $isp(2,\mathbb{R})$
$$[\eta_{X1},\eta_\mu]_\star = \frac{1}{2\sqrt{2}}\epsilon_{\mu\nu}\eta_\nu, etc..., \quad [\eta_M,\eta_N]_\star = C_{MN}^P\eta_P, \quad M = \mu, a = 1,2,3. \quad (25)$$

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- Once the Lie algebra of derivations has been choosen, simple application to the general machinery yields curvatures. Compared to the simplest situation: the pattern of covariant coordinates $\mathcal{A}_M$ larger. New derivations act as associated to "internal coordinates".

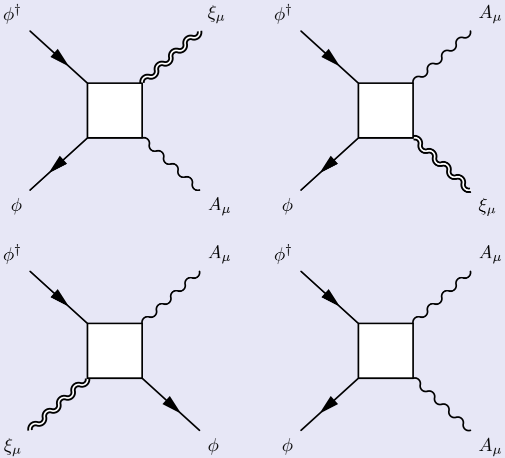
Yang-Mills-Higgs type models

- Curvature has new terms beyond $F_{\mu\nu}$. Call $\mathcal{A}_a{=}\Phi_a$, $a{=}1,2,3$.

$$F_{\mu a} = [\mathcal{A}_\mu, \Phi_a]_\star - \mu C^\nu_{\mu a}\mathcal{A}_\nu, \quad F_{ab} = [\Phi_a, \Phi_b]_\star - \mu C^c_{ab}\Phi_c \tag{26}$$

- When plugged into an action $\sim\int dx F_{MN}F_{MN}$, the second can be viewed as a Higgs potential: Higgs role played by those $\mathcal{A}_a$. The (first term)^2 involves a mass term for the gauge potential.
- Can be interpreted as Yang-Mills-Higgs type models on Moyal spaces.
- Additional couplings of the type $A_\mu\Phi\Phi$ and $A_\mu A_\mu\Phi\Phi$ that should in principle contribute to the singular part of the polarisation tensor, to be computed.

Vertices involving A_μ



Tadpole diagram I

The amplitude for the tadpole diagram is

$$\mathcal{T}_1 = \frac{\Omega^2}{4\pi^6\theta^6} \int d^4x\ d^4u\ d^4z \int_0^\infty \frac{dt\ e^{-tm^2}}{\sinh^2(\widetilde{\Omega}t)\cosh^2(\widetilde{\Omega}t)}\ A_\mu(u)\ e^{-i(u-x)\wedge z}$$
$$\times\ e^{-\frac{\widetilde{\Omega}}{4}(\coth(\widetilde{\Omega}t)z^2+\tanh(\widetilde{\Omega}t)(2x+z)^2}((1-\Omega^2)(2\widetilde{x}_\mu+\widetilde{z}_\mu)-2\widetilde{u}_\mu)$$

Introduce the following 8-dimensional vectors X, J and the 8×8 matrix K defined by

$$X=\begin{pmatrix} x \\ z \end{pmatrix},\quad K=\begin{pmatrix} 4\tanh(\widetilde{\Omega}t)\mathbb{I} & 2\tanh(\widetilde{\Omega}t)\mathbb{I}-2i\Theta^{-1} \\ 2\tanh(\widetilde{\Omega}t)\mathbb{I}+2i\Theta^{-1} & (\tanh(\widetilde{\Omega}t)+\coth(\widetilde{\Omega}t))\mathbb{I} \end{pmatrix},\quad J=\begin{pmatrix} 0 \\ i\tilde{u} \end{pmatrix}.$$

This permits one to reexpress the amplitude in a form such that some Gaussian integrals can be easily performed:

$$\mathcal{T}_1 = \frac{\Omega^2}{4\pi^6\theta^6} \int d^4x\ d^4u\ d^4z \int_0^\infty \frac{dt\ e^{-tm^2}}{\sinh^2(\widetilde{\Omega}t)\cosh^2(\widetilde{\Omega}t)}\ A_\mu(u)$$
$$\times\ e^{-\frac{1}{2}X.K.X+J.X}((1-\Omega^2)(2\widetilde{x}_\mu+\widetilde{z}_\mu)-2\widetilde{u}_\mu)$$

By performing the Gaussian integrals on X, we find

$$\mathcal{T}_1 = -\frac{\Omega^4}{\pi^2\theta^2(1+\Omega^2)^3} \int d^4u \int_0^\infty \frac{dt\ e^{-tm^2}}{\sinh^2(\widetilde{\Omega}t)\cosh^2(\widetilde{\Omega}t)}\ A_\mu(u)\widetilde{u}_\mu\ e^{-\frac{2\Omega}{\theta(1+\Omega^2)}\tanh(\widetilde{\Omega}t)u^2}.$$

Tadpole diagram II

Inspection of the behaviour of $\mathcal{T}_1$ for $t \to 0$ shows that this latter expression has a quadratic as well as a logarithmic UV divergence. From Taylor expansion:

$$\mathcal{T}_1 = -\frac{\Omega^2}{4\pi^2(1+\Omega^2)^3}\left(\int d^4u\ \widetilde{u}_\mu A_\mu(u)\right)\ \frac{1}{\epsilon}\ -\frac{m^2\Omega^2}{4\pi^2(1+\Omega^2)^3}\left(\int d^4u\ \widetilde{u}_\mu A_\mu(u)\right)\ \ln$$
$$-\frac{\Omega^4}{\pi^2\theta^2(1+\Omega^2)^4}\left(\int d^4u\ u^2\widetilde{u}_\mu A_\mu(u)\right)\ \ln(\epsilon)\ + \dots,$$

where $\epsilon \to 0$ is a cut-off and the ellipses denote finite contributions.

Higher order terms

- The regularisation of the diverging amplitudes is performed in a way that preserves gauge invariance of the most diverging terms. In $D=4$, these are UV quadratically diverging so that the cut-off ϵ on the various integrals over the Schwinger parameters ($\int_{\epsilon}^{\infty} dt$) must be suitably chosen.
- We find that this can be achieved with $\int_{\epsilon}^{\infty} dt$ for $\mathcal{T}_2''$ while for $\mathcal{T}_2'$ the regularisation must be performed with $\int_{\epsilon/4}^{\infty}$.
- In field-theoretical language, gauge invariance is broken by the naive ϵ-regularisation of the Schwinger integrals and must be restored by adjusting the regularisation scheme. Note that the logarithmically divergent part is insensitive to a finite scaling of the cut-off.

Higher order terms II

- The one-loop effective action can be expressed in terms of heat kernels:

$$\Gamma_{1loop}(\phi, A) = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{Tr}\big(e^{-tH(\phi,A)} - e^{-tH(0,0)}\big) \tag{27}$$
$$= -\frac{1}{2}\lim_{s\to 0}\Gamma(s)\,\mathrm{Tr}\big(H^{-s}(\phi,A) - H^{-s}(0,0)\big),$$

where $H(\phi, A) = \frac{\delta^2 S(\phi,A)}{\delta\phi\,\delta\phi^\dagger}$. Expanding:

$$H^{-s}(\phi, A) = \Big(1 + a_1(\phi,A)s + a_2(\phi,A)s^2 + \dots\Big)H^{-s}(0,0), \tag{28}$$

we obtain

$$\Gamma_{1loop}(\phi, A) = -\frac{1}{2}\lim_{s\to 0}\mathrm{Tr}\Big(\big(\Gamma(s+1)a_1(\phi,A) + s\Gamma(s+1)a_2(\phi,A) + \dots\big)H^{-s}(0,0)\Big).$$

With $\Gamma(s+1) = 1 - s\gamma + \dots$ we have

$$\Gamma_{1loop}(\phi, A) = -\frac{1}{2}\lim_{s\to 0}\mathrm{Tr}\big(a_1(\phi,A)H^{-s}(0,0)\big)$$
$$-\frac{1}{2}\mathrm{Res}_{s=0}\,\mathrm{Tr}\Big(\big(a_2(\phi,A) - \gamma a_1(\phi,A)\big)H^{-s}(0,0)\Big). \tag{29}$$

The second line is the Wodzicki residue which corresponds to the logarithmically divergent part of the one-loop effective action. The quadratically divergent part $-\frac{1}{2}\lim_{s\to 0}\mathrm{Tr}\big(a_1 H^{-s}(0,0)\big)$ in the action which cannot be gauge-invariant.