Derivation algebras and noncommutative gauge theories

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Content

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- Extension of the GW solution to the case of gauge theories appeared to be difficult to obtain, due to constraints from gauge invariance. This was finally achieved in 2007 [de Goursac, JCW, Wulkenhaar], confirmed after [Grosse, Wohlgenannt].

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$$S = \int d^4x \Big(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega^2}{4} \{X_\mu, X_\nu\}_\star^2 + \kappa X_\mu \star X_\mu \Big) \qquad (1)$$

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- Action has interesting features. But, unexpectedly, it has a non trivial vacuum whose expression must be determined to built the perturbative expansion. Hard to determine due to the nonlocal nature of the interaction contributions in the equations of motion. This has been carried out very recently [de Goursac, JCW, Wulkenhaar].Notice non trivial vacuum configurations also occur in NC scalar models with harmonic term as shown in [deGoursac, Tanasa, JCW].

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- Usefull algebraic framework describing simply most of the features of NCFT on Moyal spaces considered so far: differential calculus based on the derivations. Basically, concept of vector fields replaced by the derivations of the associative algebra modeling the NC space. Then, a differential calculus can be constructed provided the derivations forms a Lie algebra.

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- For Moyal spaces, the "minimal" derivation-based differential calculus generated by the ∂_μ underlies (most of) the works that appear in the litterature, so far. This differential calculus is not unique but can be modified. A simple modification permits one to interpret the (additional) covariant coordinates as Higgs field and to built naturally Yang-Mills-Higgs models [Cagnache,, Masson, JCW].

The noncommutative algebraic set-up

Derivation-based differential calculus

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- $\mathbb{A}$: associative (involutive) algebra with unit $\mathbf{1}$ and involution $a \mapsto a^*$, $\forall a \in \mathbb{A}$.
  - $\text{Der}(\mathbb{A})$, linear space of derivations of $\mathbb{A}$: space of linear maps $\mathfrak{X} : \mathbb{A} \to \mathbb{A}$, $\mathfrak{X}(ab) = \mathfrak{X}(a)b + a\mathfrak{X}(b)$, $\forall\ a, b \in \mathbb{A}$.
  - $\text{Der}(\mathbb{A})$ is a Lie algebra when equipped with internal antisym product $(\mathfrak{X}, \mathfrak{Y}) \mapsto [\mathfrak{X}, \mathfrak{Y}]$, $[\mathfrak{X}, \mathfrak{Y}](a) \equiv \mathfrak{X}(\mathfrak{Y}(a)) - \mathfrak{Y}(\mathfrak{X}(a))$, $\forall\ a \in \mathbb{A}$
  - $\text{Der}(\mathbb{A})$ can be given a module structure over $\mathcal{Z}(\mathbb{A})$, the center of $\mathbb{A}$ (Define product $\mathcal{Z}(\mathbb{A}) \times \text{Der}(\mathbb{A}) \to \text{Der}(\mathbb{A})$, $(f\mathfrak{X})(a) \equiv f\mathfrak{X}(a)$, $\forall\ f \in \mathcal{Z}(\mathbb{A})$, $\mathfrak{X} \in \text{Der}(\mathbb{A})$)

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- Inner derivation: Any $a \in \mathbb{A}$ defines a derivation $b \mapsto [a, b]_{\mathbb{A}}$ (commutator in $\mathbb{A}$!). Int($\mathbb{A}$): a Lie ideal and a $\mathcal{Z}(\mathbb{A})$-submodule in Der($\mathbb{A}$). Outer derivations: Out($\mathbb{A}$)$\equiv$Der($\mathbb{A}$)/Int($\mathbb{A}$) (Lie algebra and a $\mathcal{Z}(\mathbb{A})$-module).

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- Natural generalization of commutative case. When $\mathbb{A} = C^\infty(M)$ (M smooth manifold) , $\mathcal{Z}(\mathbb{A}) = C^\infty(M)$. No inner derivations $\mathrm{Int}(\mathbb{A}) = 0$, while $\mathrm{Der}(\mathbb{A}) = \Gamma(M)$, the Lie algebra of smooth vector fields and $\mathrm{Out}(\mathbb{A}) = \Gamma(M)$.

Noncommutative connection and curvature

- From any Lie subalgebra $\mathcal{G}\subset\text{Der}(\mathbb{A})$ (also a $\mathcal{Z}(\mathbb{A})$-submodule), construction of a differential calculus can be performed. [Space of 0-forms identified with $\mathbb{A}$, action of the differential d on 0-forms and 1-forms ($\mathcal{Z}(\mathbb{A})$-linear maps from $\mathcal{G}$ to $\mathbb{A}$) defined $\forall X, Y \in \mathcal{G}$ by $d\omega_0(X)=X(\omega_0)$, $d\omega_1(X,Y) = X(\omega_1(Y)) - Y(\omega_1(X)) - \omega_1([X,Y])$(ii). $d^2 = 0$ thanks to (i) and (ii). Can be extended to n-forms,$\mathcal{Z}(\mathbb{A})$-multilinear antisymmetric maps from $\mathcal{G}$ to $\mathbb{A}$.]

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- Once $\mathbb{A}$ equipped with diff. calculus related to $\mathcal{G}\subset\text{Der}(\mathbb{A})$, construction of NC connections and curvatures can be done [see: Connes, Dubois-Violette, Kerner, Madore]. Choose some $\mathbb{A}$-module $\mathcal{H}$.

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- NC connection on $\mathcal{H}$: a linear map $\widehat{\nabla}_{\mathfrak{X}} : \mathcal{H} \to \mathcal{H}$ such that $\forall \mathfrak{X}, \mathfrak{Y} \in \mathrm{Der}(\mathbb{A})$, $\forall a \in \mathbb{A}$, $\forall m \in \mathcal{H}$, $\forall f \in \mathcal{Z}(\mathbb{A})$:

$$\widehat{\nabla}_{\mathfrak{X}}(ma) = m(\mathfrak{X}a) + (\widehat{\nabla}_{\mathfrak{X}} m)a, \quad \widehat{\nabla}_{f\mathfrak{X}} m = f\widehat{\nabla}_{\mathfrak{X}} m, \quad \widehat{\nabla}_{\mathfrak{X}+\mathfrak{Y}} m = \widehat{\nabla}_{\mathfrak{X}} m + \widehat{\nabla}_{\mathfrak{Y}} m \tag{2}$$

Curvature of $\widehat{\nabla}$: linear map defined by

$$\widehat{R}(\mathfrak{X}, \mathfrak{Y})m = [\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}]m - \widehat{\nabla}_{[\mathfrak{X},\mathfrak{Y}]} m$$

This expression measures the departure of $\mathfrak{X} \mapsto \widehat{\nabla}_{\mathfrak{X}}$ to be a morphism of Lie algebras. This is an adaptation of the ordinary definition of connections of commutative case.

The free module case

- Hermitean structure is a sesquilinear map $h : \mathcal{H} \times \mathcal{H} \to \mathbb{A}$ such that († is the involution in $\mathbb{A}$)

$$h(m_1 a_1, m_2 a_2) = a_1^\dagger h(m_1, m_2) a_2, \quad \forall a_1, a_2 \in \mathbb{A}, \quad \forall m_1, m_2 \in \mathcal{H} \tag{3}$$

A connection is hermitean if

$$\mathfrak{X}(h(m_1, m_2)) = h(\nabla_{\mathfrak{X}}(m_1), m_2) + h(m_1, \nabla_{\mathfrak{X}}(m_2)), \quad \forall m_1, m_2 \in \mathcal{H}, \quad \forall \mathfrak{X} \in \mathcal{G} \tag{4}$$

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- Simplest and most widely studied case: $\mathcal{H}$=$\mathbb{A}$. Then, the connection is entirely determined by its action $\nabla_{\mathfrak{X}}(\mathbf{1})$ on the unit $\mathbf{1}$∈$\mathbb{A}$. Indeed,

$$\nabla_{\mathfrak{X}}(a) = \nabla_{\mathfrak{X}}(\mathbf{1})a + \mathfrak{X}(a), \quad \forall a \in \mathbb{A}, \quad \forall \mathfrak{X} \in \mathcal{G} \tag{5}$$

  $\nabla_{\mathfrak{X}}(\mathbf{1})$ serves to define a noncommutative analog of a gauge potential. When $\mathcal{H}$=$\mathbb{A}$, a convenient hermitian structure is given by $h_0(a_1, a_2) = a_1^\dagger \star a_2$, ensuring that the connections are hermitean provided $(\nabla_{\mathfrak{X}}(\mathbf{1}))^\dagger = -\nabla_{\mathfrak{X}}(\mathbf{1})$.

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- Gauge transformations γ: automorphisms of the module $\mathbb{A}$[1] preserving the hermitian structure h, $\gamma \in Aut_h(\mathbb{A})$.

$$\gamma(\mathbf{1}) \in \mathcal{U}(\mathbb{A}). \tag{6}$$

i): $\gamma(a) = \gamma(\mathbf{1}a) = \gamma(\mathbf{1})a$, $\forall a \in \mathbb{A}$ and ii): $h\big(\gamma(a_1), \gamma(a_2)\big) = h(a_1, a_2)$, $\forall a_1, a_2 \in \mathbb{A}$. Then $\gamma(\mathbf{1})^\dagger \gamma(\mathbf{1}) = \mathbf{1}$.

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- The action of $\mathcal{U}(\mathbb{A})$ on $\nabla_{\mathfrak{x}}$ is

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- Set $g \equiv \gamma(\mathbf{1})$.

$$(\nabla_{\mathfrak{X}})^{g}(\mathbf{1}) = g\nabla_{\mathfrak{X}}(\mathbf{1})g^{\dagger} + g\mathfrak{X}(g^{\dagger}), \quad \forall g \in \mathcal{U}(\mathbb{A}), \quad \forall \mathfrak{X} \in \mathcal{G} \tag{8}$$

$$(R_{(\mathfrak{X},\mathfrak{Y})}(m))^{g} = gR_{(\mathfrak{X},\mathfrak{Y})}(m)g^{\dagger} \tag{9}$$

Connections and curvatures on Moyal spaces

The Moyal algebra

- Associative unital involutive Moyal algebra:
$$\mathcal{M} = \mathcal{L} \cap \mathcal{R}. \tag{10}$$
$\mathcal{L}$ (resp. $\mathcal{R}$) the subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ whose $\star$-(noncommutative) multiplication (denoted from now by $\star$) from right (resp. left) by any Schwartz functions in $\mathbb{R}^{2n}$ is a subspace of $\mathcal{S}(\mathbb{R}^{2n})$.

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- The Moyal $\star$-product conveniently defined as

$$(a \star b)(x) = \frac{1}{(\pi\theta)^{2n}} \int d^{2n}y d^{2n}z \ a(x+y)b(x+z)e^{-i2y\Theta^{-1}z} \tag{11}$$

  on $\mathcal{S} \times \mathcal{S}$. $\Theta = \theta\Sigma$, Σ=diag$(J, ..., J)$, J given by $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

  [For details, see Gracia-Bondia, Varilly]

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- Useful relations $\quad [x_\mu, x_\nu]_\star = i\Theta_{\mu\nu}$
$$\partial_\mu(a \star b) = \partial_\mu a \star b + a \star \partial_\mu b, \qquad (a \star b)^\dagger = b^\dagger \star a^\dagger \tag{12a}$$
$$[x_\mu, a]_\star = i\Theta_{\mu\nu}\partial_\nu a, \qquad \int d^{2n}x \ (f \star h)(x) = \int d^{2n}x \ f(x).h(x) \tag{12b}$$

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where $\eta_{\mathfrak{X}}\in\mathcal{M}$.
- $$[\mathfrak{X}_1,\mathfrak{X}_2](a)=[\eta_{[\mathfrak{X}_1,\mathfrak{X}_2]},a]_\star=[[\eta_{\mathfrak{X}_1},\eta_{\mathfrak{X}_2}]_\star,a]_\star,\quad \eta_{\mathfrak{X}_1},\eta_{\mathfrak{X}_2}\in\mathcal{M},\quad \forall a\in\mathcal{M} \tag{14}$$
so that $(\eta_{[\mathfrak{X}_1,\mathfrak{X}_2]}-[\eta_{\mathfrak{X}_1},\eta_{\mathfrak{X}_2}]_\star)\in\mathcal{Z}(\mathcal{M})$.

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$$\mathfrak{X}(a)=[\eta_{\mathfrak{X}},a]_\star,\quad \forall a\in\mathcal{M} \tag{13}$$
where $\eta_{\mathfrak{X}}\in\mathcal{M}$.
- $$[\mathfrak{X}_1,\mathfrak{X}_2](a)=[\eta_{[\mathfrak{X}_1,\mathfrak{X}_2]},a]_\star=[[\eta_{\mathfrak{X}_1},\eta_{\mathfrak{X}_2}]_\star,a]_\star,\quad \eta_{\mathfrak{X}_1},\eta_{\mathfrak{X}_2}\in\mathcal{M},\quad \forall a\in\mathcal{M} \tag{14}$$
so that $(\eta_{[\mathfrak{X}_1,\mathfrak{X}_2]}-[\eta_{\mathfrak{X}_1},\eta_{\mathfrak{X}_2}]_\star)\in\mathcal{Z}(\mathcal{M})$.
- There exist canonical gauge invariant connections. Similar to what occurs in matrix models [Dubois-Violette, Kerner, Madore]. It is defined by
$$\nabla^{inv}_{\mathfrak{X}}(\mathbf{1})=-\eta_{\mathfrak{X}},\quad \forall\mathfrak{X}\in\mathcal{G} \tag{15}$$
One has
$$\nabla^{inv}_{\mathfrak{X}}(a)=\nabla^{inv}_{\mathfrak{X}}(\mathbf{1})\star a+[\eta_{\mathfrak{X}},a]_\star=-a\star\eta_{\mathfrak{X}},\quad \forall\mathfrak{X}\in\mathcal{G},\quad \forall a\in\mathcal{M} \tag{16}$$
Invariance can be verified by combining general definitions:
$$(\nabla^{inv}_{\mathfrak{X}})^\gamma(a)=-g\star(g^\dagger\star a\star\eta_{\mathfrak{X}})=-a\star\eta_{\mathfrak{X}}=\nabla^{inv}_{X}(a),\quad \forall\mathfrak{X}\in\mathcal{G},\quad \forall a\in\mathcal{M} \tag{17}$$

Properties and consequences II

- Gauge invariant connection permits one to define natural tensor forms $X_{\mathfrak{X}}$:

$$(\nabla_{\mathfrak{X}} - \nabla_{\mathfrak{X}}^{inv})(a) \equiv X_{\mathfrak{X}} \star a = (\nabla_{\mathfrak{X}}(\mathbf{1}) + \eta_{\mathfrak{X}}) \star a, \quad \forall \mathfrak{X} \in \mathcal{G}, \quad \forall a \in \mathcal{M} \qquad (18)$$

$$(X_{\mathfrak{X}})^{\gamma} = g \star X_{\mathfrak{X}} \star g^{\dagger}, \quad \forall g \in \mathcal{U}(\mathcal{M}) \qquad (19)$$

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- By using $(\eta_{[\mathfrak{X}_1,\mathfrak{X}_2]_D} - [\eta_{\mathfrak{X}_1}, \eta_{\mathfrak{X}_2}]_\star) \in \mathcal{Z}(\mathcal{M})$, curvature

$$F_{(\mathfrak{X},\mathfrak{Y})}(a) = ([X_{\mathfrak{X}}, X_{\mathfrak{Y}}]_\star - X_{[\mathfrak{X},\mathfrak{Y}]} - ([\eta_{\mathfrak{X}}, \eta_{\mathfrak{Y}}]_\star - \eta_{[\mathfrak{X},\mathfrak{Y}]})) \star a \tag{20}$$

Notice that $F^{inv}_{\mathfrak{X},\mathfrak{Y}}(a) = -([\eta_{\mathfrak{X}}, \eta_{\mathfrak{Y}}]_\star - \eta_{[\mathfrak{X},\mathfrak{Y}]}) \star a$ corresponds to the curvature for the canonical invariant connections.

Gauge theories on Moyal spaces

The minimal derivation algebra

- The simplest case: related to the spatial ∂_μ. Set $\nabla_\mu(\mathbf{1}) \equiv -iA_\mu$

$$\partial_\mu a = [i\xi_\mu, a]_\star, \quad \xi_\mu \equiv -\Theta^{-1}_{\mu\nu}x^\nu, \quad \forall a \in \mathcal{M} \tag{21}$$

$$\eta_\mu = i\xi_\mu, \quad \mu = 1, ..., D(=2n) \tag{22}$$

Covariant coordinates:

$$X_\mu = -i(A_\mu - \xi_\mu) \tag{23}$$

$$F_{\mu\nu} = [X_\mu, X_\nu]_\star - i\Theta^{-1}_{\mu\nu} = -i\big(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star\big) \tag{24}$$

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- NC analog of Yang-Mills action $\int d^4x(F_{\mu\nu} \star F_{\mu\nu})(x)$ has UV/IR mixing which spoilts renormalisability.
- Examine how to extend the Grosse-Wulkenhaar solution that give rise to renormalisable scalar theories to the case of gauge theories? The 1st problem is to extend the Grosse-Wulkenhaar harmonic term so that gauge invariance of the action is preserved.

Gauge theory models on Moyal spaces

- In $D = 4$, the result is (De Goursac, JCW, Wulkenhaar)

$$S = \int d^4x \Big(\frac{\alpha}{4g^2} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega'}{4g^2} \{X_\mu, X_\nu\}_\star^2 + \frac{\kappa}{2} X_\mu \star X_\mu \Big)$$

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- Appears to be related possibly to a spectral triple (Grosse, Wulkenhaar).
- the X_μ's are reminiscent of Higgs field
- Problem:S has a non trivial vacuum whose explicit expression must be determined in order to define perturbative expansion. Very hard to determine due to the nonlocal nature of the interaction contributions in the equations of motion.

Vacuum configurations

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- Convenient to represent elements on $\mathcal{M}$ in the matrix base. Indeed, for all $g \in \mathcal{M}$, there is a unique matrix (g_{mn}) satisfying

$$\forall x \in \mathbb{R}^D \quad g(x) = \sum_{m,n \in \mathbb{N}^{\frac{D}{2}}} g_{mn} b_{mn}^{(D)}(x). \tag{25}$$

The matrix base satisfies to the following properties:

$$(b_{mn}^{(D)} \star b_{kl}^{(D)})(x) = \delta_{nk} b_{ml}^{(D)}(x), \tag{26}$$

$$\int d^D x \; b_{mn}^{(D)}(x) = (2\pi\theta)^{\frac{D}{2}} \delta_{mn}, \tag{27}$$

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- In D=2, $(b^{(2)}_{mn}) = (f_{mn})$ of the matrix base in polar coordinates $x_1 = r\cos(\varphi)$, $x_2 = r\sin(\varphi)$ is

$$f_{mn}(x) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i(n-m)\varphi} \left(\frac{2r^2}{\theta}\right)^{\frac{n-m}{2}} L^{n-m}_m \left(\frac{2r^2}{\theta}\right) e^{-\frac{r^2}{\theta}}, \tag{29}$$

In $D = 4$, $m = (m_1, m_2)$, $n = (n_1, n_2)$ and $b^{(4)}_{mn}(x) = f_{m_1,n_1}(x_1, x_2)\ f_{m_2,n_2}(x_3, x_4)$

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Vacuum configurations

- Start from

$$S=\int d^Dx\ \Big(\frac{1}{4}F_{\mu\nu}\star F_{\mu\nu}+\frac{\Omega^2}{4}\{X_\mu,X_\nu\}_\star^2+\kappa X_\mu\star X_\mu\Big) \tag{31}$$

$$-2(1-\Omega^2)X_\nu\star X_\mu\star X_\nu+(1+\Omega^2)X_\mu\star X_\nu\star X_\nu+$$
$$(1+\Omega^2)X_\nu\star X_\nu\star X_\mu+2\kappa X_\mu=0 \tag{32}$$

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- Look for solution of the form $X_\mu(x) = \Phi_1(x^2) x_\mu + \Phi_2(x^2) \widetilde{x}_\mu$ (we set $\widetilde{x}_\mu = 2\Theta^{-1}_{\mu\nu} x_\nu$). In D=2 and D=4, combine this to the matrix base to reduce $\star$-products to matrix products. Then check that it is a minimum for the action. Finally, go back to the x-space by using

$$L^k_m(z) = \frac{e^z z^{-\frac{k}{2}}}{m!} \int_0^\infty dt \ e^{-t} t^{m+\frac{k}{2}} J_k(2\sqrt{tz}) \tag{33}$$

and

$$\sum_{k=0}^{m} L^\alpha_k(x) L^\beta_{m-k}(y) = L^{\alpha+\beta+1}_m(x+y) \tag{34}$$

Vacuum configurations

- Final solutions:

$$X_\mu^{2D}(x) = 2\sqrt{\theta}\frac{e^{\frac{z}{2}}}{\sqrt{z}}\int_0^\infty dt\ e^{-t}\sqrt{t}J_1(2\sqrt{tz})$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m\sqrt{u_{m+1}}}{m!\sqrt{m+1}}t^m\left(\widetilde{x}_\mu\cos(\xi_m)+\frac{2}{\theta}x_\mu\sin(\xi_m)\right) \tag{35}$$

$$X_\mu^{4D}(x) = 2\sqrt{2\theta}\frac{e^{\frac{z}{2}}}{z}\int_0^\infty dt\ e^{-t}tJ_2(2\sqrt{tz})$$

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where $z=\frac{2x^2}{\theta}$, the ξ'are real, u_m and v_m functions of m and the parameters in the action.

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- Next step: Determine salient features (if any) of these vacuum configuration; expansion of the action around the vacuum. Combined use of x-space formulation and matrix base may be helpful.

Yang-Mills-Higgs actions on Moyal spaces

- Initial motivation: Examine if "covariant coordinates" can be interpreted as Higgs fields. Needs to disentangle covariant coordinates from gauge potential. Try to enlarge the minimal algebra of derivations on $\mathcal{M}$. Since all derivations are inner, to each new derivation corresponds a new covariant coordinate (tensor form).

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- The Moyal bracket reduces to the Poisson bracket when restricted to polynomials of degree less or equal than 2: for a and b, polynomials of degree ≤ 2, $[a,b]_\star = i\Theta_{\mu\nu}\frac{\partial a}{\partial x^\mu}\frac{\partial b}{\partial y^\nu} \equiv \{a,b\}_{PB}$.

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- Supplement the $\partial_\mu a = [i\xi_\mu, a]_\star (\xi_\mu \equiv -\Theta^{-1}_{\mu\nu}x^\nu)$ generated by degree 1 polynomials by those derivations corresponding to degree 2 polynomials [Cagnache, Masson, JCW]. The $\frac{D(D+1)}{2}$ monomials $(x_\mu x_\nu)$ correspond to new derivations η_X supplementing the spatial derivations.

Yang-Mills-Higgs actions on Moyal spaces

- The η_{X_a}, $a = 1, 2, ..., \frac{D(D+1)}{2}$, satisfy the commutation relations for $sp(D, \mathbb{R})$. We set
$$[\eta_{X_M}, \eta_{X_N}]_\star = \mu C^P_{MN} \eta_{X_P} \tag{37}$$
Mass parameter μ insures that the various objects have suitable mass dimensions, and the C^P_{MN} are the dimensionless structure constants.Convenient to set: capital Latin letters M, N covering both the spatial and additional directions, respectively denoted by greek indices μ, ν,...$= 1, 2, ..., D$, and Latin indices a, b,...$=$ $1, 2, ..., \frac{D(D+1)}{2}$. Derivations are then labeled as X_M ($X_\mu = \partial_\mu, X_a$).

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- Components of the curvature obtained by applying the general algebraic framework
$$F_{\mu\nu}=[X_\mu,X_\nu]_\star-i\Theta^{-1}_{\mu\nu}=-i(\partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu,A_\nu]_\star) \tag{38}$$
$$F_{\mu a}=[X_\mu,X_a]_\star-\mu C_{\mu a}^\nu X_\nu=\partial_\mu X_a-i[A_\mu,X_a]_\star-\mu C_{\mu a}^\nu X_\nu \tag{39}$$
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- From $\int d^D x F_{MN} \star F_{MN}$, one obtains a Yang-Mills-Higgs model: F_{ab}^2 is the quartic Higgs potential, $F_{\mu a}^2$ involves kinetic term for X_a. By inspection of the various terms, each additional inner derivation (which may be viewed as related to an extra (noncommutative) dimension) corresponds to an additional covariant coordinate that can be interpreted as a Higgs field.

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- Can be viewed as extension of the old matrix-valued models Dubois-Violette, Kerner, Madore to the case of NC Moyal space.
- Those models still have UV/IR mixing in any D-dimensions.