Statistics of the number of zero crossings: from diffusion equation to random polynomials

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Introduction

Persistence probability \( p_0(t) \)

- \( X(t) \equiv \) stochastic random variable evolving in time \( t \), \( \langle X(t) \rangle = 0 \)
- Persistence probability
  \[ p_0(t) \equiv \text{Proba. that } X \text{ has not changed sign up to time } t \]
Persistence probability $p_0(t)$

- $X(t) \equiv$ stochastic random variable evolving in time $t$, $\langle X(t) \rangle = 0$
- Persistence probability
  
  \[
p_0(t) \equiv \text{Proba. that } X \text{ has not changed sign up to time } t
  \]

Persistence in physical systems

- coarsening spin field at low $T$
- height of a fluctuating interface
- diffusion field

\[
p_0(t) \propto t^{-\theta_p}
\]
Motivations: persistence for the diffusion equation

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]
\[ \langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x') \]

Diffusion equation with random initial conditions

Single length scale
\[ \ell(t) \propto t^{1/2} \]
Motivations: persistence for the diffusion equation

The diffusion equation with random initial conditions is given by:

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]

With initial conditions:

\[ \langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x') \]

Single length scale:

\[ \ell(t) \propto t^{1/2} \]

Persistence \( p_0(t, L) \) for a \( d \)-dim. system of linear size \( L \):

\[ p_0(t, L) \equiv \text{Proba. that } \phi(x, t) \text{ has not changed sign up to } t \]

S. N. Majumdar, C. Sire, A. J. Bray and S. J. Cornell, PRL 96

B. Derrida, V. Hakim and R. Zeitak, PRL 96
Motivations: persistence for the diffusion equation

Diffusion equation with random initial conditions

$$\partial_t \phi(x, t) = \nabla^2 \phi(x, t)$$

$$\langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x')$$

Single length scale

$$\ell(t) \propto t^{1/2}$$

Persistence $p_0(t, L)$ for a $d$-dim. system of linear size $L$

$$p_0(t, L) \equiv \text{Proba. that } \phi(x, t) \text{ has not changed sign up to } t$$

 Persistence $p_0(t, L)$ for a $d$-dim. system of linear size $L$. The probability $p_0$ of not changing sign up to time $t$ is shown for different linear sizes $L = 32, 64, 128, 256$. The graph demonstrates the decay of $p_0$ with time, scaling as $t^{-\theta(d)}$. The exponent $\theta(d)$ depends on the dimension $d$. The graph also shows the scaling of the single length scale $\ell(t)$ with $t^{1/2}$. The data points are plotted on a log-log scale for clarity.
Motivations: persistence for the diffusion equation

Diffusion equation with random initial conditions

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]
\[ \langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x') \]

Single length scale
\[ \ell(t) \propto t^{1/2} \]

Persistence \( p_0(t, L) \) for a \( d \)-dim. system of linear size \( L \)

\[ p_0(t, L) \equiv \text{Proba. that } \phi(x, t) \text{ has not changed sign up to } t \]

\[ p_0(t, L) \propto L^{-2\theta(d)} h(t/L^2) \]

\[ \theta(1) = 0.1207 \]
\[ \theta(2) = 0.1875 \quad , \quad \text{Numerics} \]
Measurement of Persistence in 1D Diffusion

Glenn P. Wong, Ross W. Mair, and Ronald L. Walsworth
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David G. Cory
Department of Nuclear Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 1 September 2000)

\[ p_0(t, L) \propto t^{-\theta_{\exp}(1)} \]
\[ \theta_{\exp}(1) \approx 0.12 \]
Motivations: Real random polynomials

Real Kac's polynomials

\[ K_n(x) = \sum_{i=0}^{n-1} a_i x^i \]

\( a_i \equiv \text{Gaussian random variables,} \)
\( \langle a_i \rangle = 0, \langle a_i a_j \rangle = \delta_{ij} \)
Motivations: Real random polynomials

Real Kac’s polynomials

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Complex roots

G.Schehr (LPT Orsay)
Motivations: Real random polynomials

Real Kac’s polynomials

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\[ a_i \equiv \text{Gaussian random variables, } \langle a_i \rangle = 0, \langle a_i a_j \rangle = \delta_{ij} \]

Real roots

\[ \mathcal{N}_n \equiv \text{mean number of roots on the real axis} \quad \text{M.Kac '43} \]

\[ \mathcal{N}_n \sim \frac{2}{\pi} \log n \]
Motivations: Real roots of Kac’s polynomials

$q_0(n) \equiv \text{Probability that } K_n(x) \text{ has no real root in } [0, 1]$

$q_0(n) \propto n^{-\gamma}$

with $\gamma = 0.19(1)$ (Numerics)
Purpose: a link between random polynomials & diffusion equation

**Generalized Kac's polynomials**

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i \]

\[ a_i \equiv \text{Gaussian random variables}, \quad \langle a_i \rangle = 0, \quad \langle a_i a_j \rangle = \delta_{ij} \]

**Proba. of no real root**

\[ q_0(n) \propto n^{-b(d)} \]

**Persistence of diffusion**

\[ p_0(t, L) \propto L^{-2\theta(d)} \]
**Generalized Kac’s polynomials**

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i x^{(d-2)/4} i^i \]

\[ a_i \equiv \text{Gaussian random variables,} \quad \langle a_i \rangle = 0, \quad \langle a_i a_j \rangle = \delta_{ij} \]

**Proba. of no real root**

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**Persistence of diffusion**

\[ p_0(t, L) \propto L^{-2\theta(d)} \]

\[ b(d) = \theta(d) \]

G. S., S. N. Majumdar PRL 07
1. Mapping to a Gaussian Stationary Process (GSP)
   - Two theorems for GSP
   - The case of diffusion equation
   - The case of random polynomials
   - Numerical check
   - A heuristic argument
   - Conclusion

2. Probability of $k$-zero crossings
   - Generalization to $k$ zero crossings for diffusion and polynomials
   - Mean field approximation and large deviation function
   - A more refined analysis
   - Conclusion
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Slepian’s theorems (’62)

- $X(T) \equiv$ Gaussian Stationary Process
  \[
  \langle X(T) \rangle = 0, \quad \langle X(T)X(T') \rangle = f(|T - T'|)
  \]
- $P_0(T) \equiv$ Proba. that $X$ has not change sign up to time $T$

**Theorem n°1**

\[
f(T) = \exp(-\lambda T) \implies P_0(T) \sim \exp(-\lambda T), \ T \gg 1
\]
Slepian’s theorems ('62)

- $X(T) \equiv$ Gaussian Stationary Process
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**Theorem n°1**

\[ f(T) = \exp(-\lambda T) \implies P_0(T) \sim \exp(-\lambda T), \quad T \gg 1 \]

**Theorem n°2**

\[ f(T) \sim \exp(-\lambda T), \quad T \gg 1 \implies P_0(T) \sim \exp(-\mu T), \quad T \gg 1 \]
Outline

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Persistence of diffusion equation

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]
\[ \langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x') \]

\[ \phi(x, t) = \int_{|y|<L} d^d y G(x - y, t) \phi(y, 0) \]

\[ G(x, t) = (4\pi t)^{-\frac{d}{2}} \exp \left( -\frac{x^2}{4t} \right) \]
Persistence of diffusion equation

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]

\[ \langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x') \]

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Mapping of \( \phi(x, t) \) to a Gaussian stationary process

1. Normalized process \( X(t) = \frac{\phi(x,t)}{(\langle \phi(x,t)^2 \rangle)^{1/2}} \)

\[ \langle X(t)X(t') \rangle \sim \begin{cases} \left( \frac{tt'}{(t+t')^2} \right)^{d/4}, & t, t' \ll L^2 \\ 1, & t, t' \gg L^2 \end{cases} \]

2. New time variable \( T = \log t \), for \( t \ll L^2 \)

\[ \langle X(T)X(T') \rangle = \left[ \cosh((T - T')/2) \right]^{-d/2} \]
Persistence for a Gaussian stationary process (GSP)

- $X(T)$ is a GSP with correlations
  \[
  \langle X(T)X(T') \rangle = a(T - T')
  \]
  \[
  a(T) = (\cosh(T/2))^{-d/2}
  \]

- Persistence probability $P_0(T)$
- Slepian's Theorem

For $T \gg 1$ \[ a(T) \propto \exp\left(-\frac{d}{2}T\right) \Rightarrow P_0(T) \propto \exp\left(-\theta(d)T\right) \]

Reverting back to $t = \exp(T)$

\[ p_0(t, L) \sim t^{-\theta(d)} \quad 1 \ll t \ll L^2 \]
Normalized process $X(t) = \frac{\phi(x,t)}{\langle \phi(x,t)^2 \rangle^{1/2}}$

$$\langle X(t)X(t') \rangle \sim \begin{cases} 
\left( \frac{4tt'}{(t+t')^2} \right)^{\frac{d}{4}}, & t, t' \ll L^2 \\
1, & t, t' \gg L^2 
\end{cases}$$
Persistence of diffusion equation

- Normalized process \( X(t) = \frac{\phi(x,t)}{\langle \phi(x,t)^2 \rangle^{1/2}} \)

\[
\langle X(t)X(t') \rangle \sim \begin{cases} 
\left(4 \frac{tt'}{(t+t')^2}\right)^\frac{d}{4}, & t, t' \ll L^2 \\
1, & t, t' \gg L^2 
\end{cases}
\]

\[
p_0(t, L) \propto L^{-2\theta(d)} h(t/L^2) \\
h(u) \sim \begin{cases} 
u \sim u^{-\theta(d)}, & u \ll 1 \\
u \sim \text{cst}, & u \gg 1 
\end{cases}
\]
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Real roots of generalized Kac’s polynomials

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i \]

Averaged density of real roots

\[ \rho_n(x) = \langle |K'_n(x)| \delta(K_n(x)) \rangle \]

Real roots concentrate around \( x = \pm 1 \)

\[ \rho_n(\pm 1) \sim A_d n \]

\[ A_d = \frac{2\sqrt{d/(d + 4)}}{\pi(d + 2)} \]
Real roots of generalized Kac’s polynomials

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i \]

Averaged density of real roots for \( n \to \infty \)

\[ \rho_\infty(x) = \frac{\left( \text{Li}_{-1-d/2}(x^2)(1 + \text{Li}_{1-d/2}(x^2)) - \text{Li}_{2-d/2}(x^2) \right)^{1/2}}{\pi |x|(1 + \text{Li}_{1-d/2}(x^2))} \]
Real roots of generalized Kac’s polynomials

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i \]

Mean number of real roots in \([0, 1]\): Kac-Rice formula

\[ \langle N_n[0, 1] \rangle = \int_0^1 \rho_n(x) \, dx \sim \frac{1}{2\pi} \sqrt{\frac{d}{2}} \log n \]
Probability of no real root for $K_n(x)$

$P_0(x, n) \equiv \text{Proba. that } K_n(x) \text{ has no real root in } [0, x]$
Probability of no real root for $K_n(x)$

- **Two-point correlator**
  
  $$C_n(x, y) = \langle K_n(x)K_n(y) \rangle = \sum_{i=0}^{n-1} i^{(d-2)/2}(xy)^i$$

- **Normalization**
  
  $$C_n(x, y) = \frac{C_n(x, y)}{(C_n(x, x))^{1/2}(C_n(y, y))^{1/2}}$$

- **Change of variable**
  
  $$x = 1 - \frac{1}{t} \quad , \quad t \gg 1$$
Probability of no real root for $K_n(x)$

Normalized correlator in the scaling limit

- **Scaling limit**

  $$t \gg 1 \ , \ n \gg 1 \ \text{keeping} \quad \tilde{t} = \frac{t}{n} \ \text{fixed}$$

- $C_n(t, t') \rightarrow C(\tilde{t}, \tilde{t}')$ with the asymptotic behaviors

  $$C(\tilde{t}, \tilde{t}') \sim \begin{cases} 
  \left( \frac{4\frac{\tilde{t}\tilde{t}'}{(\tilde{t}+\tilde{t}')^2}}{4} \right)^{\frac{d}{4}} , & \tilde{t}, \tilde{t}' \ll 1 \\
  1 , & \tilde{t}, \tilde{t}' \gg 1 
  \end{cases}$$
Persistence of diffusion equation (reminder)

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]

\[ \langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x') \]

\[ \phi(x, t) = \int d^d y G(x - y) \phi(y, 0) \]

\[ G(x, t) = (4\pi t)^{-\frac{d}{2}} \exp \left( -\frac{x^2}{4t} \right) \]

Mapping of \( \phi(x, t) \) to a Gaussian stationary process

1. Normalized process \( X(t) = \frac{\phi(x, t)}{\langle \phi(x, t)^2 \rangle^{1/2}} \)

\[ \langle X(t)X(t') \rangle \sim \begin{cases} 1, & t, t' \gg L^2 \\ \left( \frac{tt'}{(t+t')^2} \right)^{\frac{d}{4}}, & t, t' \ll L^2 \end{cases} \]

2. Persistence probability \( p_0(t, L) \)

\[ p_0(t, L) \propto L^{-2\theta(d)} h(t/L^2) \]
Probability of no real root for $K_n(x)$

$$C(\tilde{t}, \tilde{t}') \sim \begin{cases} \left(4 \frac{\tilde{t}\tilde{t}'}{(\tilde{t}+\tilde{t}')^2}\right)^{\frac{d}{4}}, & \tilde{t}, \tilde{t}' \ll 1 \\ 1, & \tilde{t}, \tilde{t}' \gg 1 \end{cases}$$

$P_0(x, n) \equiv$ Proba. that $K_n(x)$ has no real root in $[0, x]$

Scaling form for $P_0(x, n)$

$$P_0(x, n) \propto n^{-\theta(d)} \tilde{h}(n(1 - x))$$

$$\tilde{h}(u) \sim \begin{cases} c^{st}, & u \ll 1 \\ u^{\theta(d)}, & u \gg 1 \end{cases}$$
Probability of no real root for $K_n(x)$

Scaling form for $P_0(x, n)$

$$P_0(x, n) \propto n^{-\theta(d)} \tilde{h}(n(1 - x))$$

$$\tilde{h}(u) \sim \begin{cases} 
  c^{st}, & u \ll 1 \\
  u^{\theta(d)}, & u \gg 1 
\end{cases}$$

$q_0(n) \equiv$ Probability that $K_n(x)$ has no real root in $[0, 1]$

$$q_0(n) = P(1, n) \sim n^{-\theta(d)}$$
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2. Probability of $k$-zero crossings
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Numerical check of the scaling form

Numerical computation of $P_0(x, n)$ for $d = 2$

$P_0(x, n)$

$n = 128$
$n = 256$
$n = 512$
$n = 1024$
Numerical check of the scaling form

Numerical computation of $P_0(x, n)$ for $d = 2$

\[ P_0(x, n) \propto n^{-\theta(d)} \tilde{h}(n(1 - x)) \]
Numerical check of the exponent

Numerical computation of $P(1, n) \propto n^{-\theta(d)}$

In agreement with the exponent $\theta(d)$ found for diffusion
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A heuristic argument

**Diffusion equation**

\[
\phi(x = 0, t) = (4\pi t)^{-d/2} \int_{0 < |x| < L} d^d x \exp \left( -\frac{x^2}{4t} \right) \phi(x, 0)
\]

\[
= \frac{S_d^{1/2}}{(4\pi t)^{d/2}} \int_0^L dr \, r^{1/2(d-1)} e^{-\frac{r^2}{4t}} \Psi(r)
\]

\[
\Psi(r) = S_d^{-1/2} r^{-1/2(d-1)} \lim_{\Delta r \to 0} \frac{1}{\Delta r} \int_{r < |x| < r + \Delta r} d^d x \, \phi(x, 0)
\]

\[
\langle \Psi(r) \Psi(r') \rangle = \delta(r - r')
\]
A heuristic argument

Diffusion equation

\[ \phi(x = 0, t) \propto \int_0^{L^2} du \, u \frac{d-2}{4} e^{-\frac{u}{t}} \tilde{\Psi}(u) \]

\[ \langle \tilde{\Psi}(u)\tilde{\Psi}(u') \rangle = \delta(u - u') \]
A heuristic argument

- Diffusion equation

\[
\phi(x = 0, t) \propto \int_0^{L^2} du \ u \frac{d-2}{4} \ e^{-\frac{u}{t}} \tilde{\Psi}(u)
\]

\[
\langle \tilde{\Psi}(u)\tilde{\Psi}(u') \rangle = \delta(u - u')
\]

- Random polynomials: \( K_n(x) = a_0 + \sum_{i=1}^{n} a_i i \frac{d-2}{4} x^i \)

\[
K_n(1 - 1/t) \sim a_0 + \sum_{i=1}^{n} i \frac{d-2}{4} \ e^{-\frac{i}{t}} a_i
\]

\[
\sim \int_0^n du \ u \frac{d-2}{4} \ e^{-\frac{u}{t}} a(u)
\]

\[
\langle a(u)a(u') \rangle = \delta(u - u')
\]
Mapping to a Gaussian Stationary Process (GSP)
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A link between diffusion equation and random polynomials

- Proba. of no real root:
  \[ q_0(n) \propto n^{-b(d)} \]

- Persistence of diffusion:
  \[ p_0(t, L) \propto L^{-2\theta(d)} \]

- Experimental realization of real roots of real polynomials

1. Experimental realization of real roots of real polynomials

2. Towards exact results for \( \theta(d) \)
$P_0(x, n) \equiv \text{Proba. that } K_n(x) \text{ has no real root in } [0, x]$

$$P_0(x, n) \propto n^{-\theta(d)} \tilde{h}(n(1 - x))$$

- Diffusion equation appears for various other polynomials

$$W_n(x) = \sum_{i=0}^{n} a_i \frac{x^i}{\sqrt{i!}}$$

$$B_n(x) = \sum_{i=0}^{n} a_i \sqrt{\binom{n}{i}} x^i$$
Outline

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Generalization to \( k \) zero crossings

- **Diffusion equation**

  \[
  \partial_t \phi(x, t) = \nabla^2 \phi(x, t)
  \]

  \[
  \langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x')
  \]

- **Real polynomials**

  \[
  K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i
  \]

- **Probability**

  \[
  p_k(t, L) \equiv \text{Proba. that } \phi(x, t) \text{ crosses zero } k \text{ times up to } t
  \]

  \[
  q_k(n) \equiv \text{Proba. that } K_n(x) \text{ has exactly } k \text{ real roots}
  \]
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Mean field approximation for diffusion equation

- For $t \ll L^2$, $p_k(t, L)$ is given by $\mathcal{P}_k(T)$, $T = \log t$

  \[\mathcal{P}_k(T) \equiv \text{Proba. that } X(T) \text{ crosses zero } k \text{ times up to } T\]

- $\langle X(T)X(T') \rangle = (\cosh(T - T'))^{-d/2} = 1 - \frac{d}{16} T^2 + o(T^2)$

  \[\langle \mathcal{N}(T) \rangle \equiv \text{Mean number of zero crossings in } [0, T]\]

  \[= \rho T \quad \rho = \frac{1}{2\pi} \sqrt{\frac{d}{2}}\]

- Assuming that the zeros of $X(t)$ are independent

  \[\mathcal{P}_k(T) = \frac{(\rho T)^k}{k!} e^{-\rho T}\]
Mean field approximation for diffusion equation

- For $t \ll L^2$, $p_k(t, L)$ is given by $P_k(T)$, $T = \log t$

  $P_k(T) \equiv \text{Proba. that } X(T) \text{ crosses zero } k \text{ times up to } T$

- Assuming that the zeros of $X(t)$ are independent

  $P_k(T) = \frac{(\rho T)^k}{k!} e^{-\rho T}$

- For $k \gg 1$, $T \gg 1$, $k/\rho T$ fixed

  $\log P_k(T) \sim -T \varphi \left( \frac{k}{\rho T} \right)$, \quad \varphi(x) = \rho(x \log x - x + 1)$
Scaling form for GSP

\[ X(T) \text{ is a GSP with } \langle X(T)X(T') \rangle = \left[ \text{sech}\left(\frac{T - T'}{2}\right) \right]^{d/2} \]

\[ \mathcal{P}_k(T) \equiv \text{Proba. that } X(T) \text{ crosses 0 exactly } k \text{ times up to } T \]

\[ \log \mathcal{P}_k(T) = -T \varphi \left( \frac{k}{\rho T} \right) \]

\( \varphi(x) \) is a large deviation function
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- Conclusion
Generating function

\[ \hat{P}(z, T) = \sum_{k=0}^{\infty} z^k P(k, T) \sim \exp(-\hat{\theta}(z) T) \]

where \( \hat{\theta}(z) \) depends \textit{continuously} on \( z \)
A more refined analysis

Generating function

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where \( \hat{\theta}(z) \) depends continuously on \( z \)

Assuming the proposed scaling form for \( P(k, T) \)

\[ \hat{P}(z, T) = \sum_{k=0}^{\infty} z^k \exp \left( -T \varphi \left( \frac{k}{\rho T} \right) \right) \]

\[ \propto \exp \left( -\min_{x > 0} [\varphi(x) - \rho x \log z] \right) \]
A more refined analysis

- Generating function

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- Inverting the Legendre transform

\[ \varphi(x) = \text{Max}_{0 \leq z \leq 2} [\rho x \log z + \hat{\theta}(z)] \]
Application to the random acceleration process

- Random acceleration process

\[ \frac{d^2 x(t)}{dt^2} = \zeta(t) \quad , \quad \langle \zeta(t) \zeta(t') \rangle = \delta(t - t') \]

- Generating function

\[ \hat{P}(z, T) = \sum_{k=0}^{\infty} z^k P(k, T) \sim \exp\left(-\hat{\theta}(z) T\right) \]

\[ \hat{\theta}(z) = \frac{1}{4} \left( 1 - \frac{6}{\pi} \arcsin \left( \frac{z}{2} \right) \right) \quad , \quad 0 \leq z \leq 2 \]
Random acceleration process

\[ \frac{d^2 x(t)}{dt^2} = \zeta(t) \quad , \quad \langle \zeta(t) \zeta(t') \rangle = \delta(t - t') \]

Exact result

\[ \log P_k(T) = -T \varphi \left( \frac{k}{\rho T} \right) , \quad \rho = \frac{\sqrt{3}}{2\pi} \]

\[ \varphi(x) = \frac{\sqrt{3}}{2\pi} x \log \left( \frac{2x}{\sqrt{x^2 + 3}} \right) + \frac{1}{4} \left( 1 - \frac{6}{\pi} \text{ArcSin} \left( \frac{x}{\sqrt{x^2 + 3}} \right) \right) \]
Probability of $k$-zero crossings for diffusion equation

\[
\frac{\partial}{\partial t} \phi(x, t) = \nabla^2 \phi(x, t)
\]

\[
\langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x')
\]

$p_k(t, L) \equiv \text{Proba. that } \phi(x, t) \text{ crosses zero } k \text{ times up to } t$

\[
p_k(t, L) \sim t^{-\varphi\left(\frac{k}{\log t}\right)}, \quad t \ll L^2
\]
Numerical check

\[- \frac{\log p_k(t, L)}{\log t} \] as a function of \( k \)

\[
\frac{-\log(p_k(t, L))}{\log(t)}
\]

\[
t=256
\]
\[
t=512
\]
\[
t=1024
\]
\[
t=2048
\]
Numerical check

\[- \log \frac{p_k(t, L)}{\log t} \text{ as a function of } \frac{k}{\log t}\]
Outline

1. Mapping to a Gaussian Stationary Process (GSP)
   - Two theorems for GSP
   - The case of diffusion equation
   - The case of random polynomials
   - Numerical check
   - A heuristic argument
   - Conclusion

2. Probability of $k$-zero crossings
   - Generalization to $k$ zero crossings for diffusion and polynomials
   - Mean field approximation and large deviation function
   - A more refined analysis
   - Conclusion
Conclusion

- Large deviation function for random polynomials

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i \]

\[ q_k(n) \equiv \text{Proba. that } K_n(x) \text{ has exactly } k \text{ real roots in } [0, 1] \]

\[ q_k(n) \propto n^{-\varphi\left(\frac{k}{\log n}\right)} \]

- Extension to other class of polynomials
- Extension to real eigenvalues of real random matrices (Ginibre’s matrices)