QUANTUM FIELD THEORY

Tutorials (n'3)

1. **Complex field.-** Let us study the Hamiltonian of the free complex field.

   (a) Show that the operators $\hat{a}_p^{(\dagger)}$ and $\hat{\tilde{a}}_p^{(\dagger)}$ satisfy the canonical commutation relations as well as $[\hat{a}_p^{(\dagger)}, \hat{\tilde{a}}_p^{(\dagger)}] = 0$, where the annihilation operator $\hat{\tilde{a}}_p$ is associated to the anti-particle of 4-momentum p^μ.

   (b) Express the Hamiltonian $\hat{H}$ in terms of the annihilation (creation) operators $\hat{a}_p^{(\dagger)}$ and $\hat{\tilde{a}}_p^{(\dagger)}$.

   Starting from the result of the course (beginning of Section 4.3):

$$\hat{H} = \frac{1}{2}\int d^3p\ E_p \left[(\hat{a}_{1p}\hat{a}_{1p}^\dagger + \hat{a}_{1p}^\dagger\hat{a}_{1p}) + (\hat{a}_{2p}\hat{a}_{2p}^\dagger + \hat{a}_{2p}^\dagger\hat{a}_{2p})\right] ,$$

   let us compare it with (omitting the p subscripts just here),

$$\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = \frac{1}{2}(\hat{a}_1\hat{a}_1^\dagger - i\hat{a}_1\hat{a}_2^\dagger + i\hat{a}_2\hat{a}_1^\dagger + \hat{a}_2\hat{a}_2^\dagger + \hat{a}_1^\dagger\hat{a}_1 + i\hat{a}_1^\dagger\hat{a}_2 - i\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) ,$$

$$\hat{\tilde{a}}\hat{\tilde{a}}^\dagger + \hat{\tilde{a}}^\dagger\hat{\tilde{a}} = \frac{1}{2}(\hat{a}_1\hat{a}_1^\dagger + i\hat{a}_1\hat{a}_2^\dagger - i\hat{a}_2\hat{a}_1^\dagger + \hat{a}_2\hat{a}_2^\dagger + \hat{a}_1^\dagger\hat{a}_1 - i\hat{a}_1^\dagger\hat{a}_2 + i\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) ,$$

   implying,

$$\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{\tilde{a}}\hat{\tilde{a}}^\dagger + \hat{\tilde{a}}^\dagger\hat{\tilde{a}} = \hat{a}_1\hat{a}_1^\dagger + \hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2\hat{a}_2^\dagger + \hat{a}_2^\dagger\hat{a}_2 ,$$

   so that,

$$\hat{H} = \frac{1}{2}\int d^3p\ E_p\ (\hat{a}_p\hat{a}_p^\dagger + \hat{a}_p^\dagger\hat{a}_p + \hat{\tilde{a}}_p\hat{\tilde{a}}_p^\dagger + \hat{\tilde{a}}_p^\dagger\hat{\tilde{a}}_p) .$$

2. **Propagator.-** Demonstrate that the propagator for the complex field operator $\hat{\phi}(x^\mu)$ obeys the property

$$iG(x^\mu - x'^\mu) \left(=< 0|\tau[\hat{\phi}(x^\mu)\hat{\phi}^\dagger(x'^\mu)]|0 >\right) = iG(x'^\mu - x^\mu) ,$$

   where x^μ denotes the 4-coordinates and τ selects the time-ordering.

Let us for instance consider the case $t' > t$ (the other case undergoes a similar demonstration):

$$\begin{aligned} iG(x'^{\mu}-x^{\mu}) &= <0|\tau[\hat{\phi}(x'^{\nu})\hat{\phi}^{\dagger}(x^{\nu})]|0>=<0|\hat{\phi}(x'^{\nu})\hat{\phi}^{\dagger}(x^{\nu})|0> \\ &= <0|\int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}(\hat{a}_p e^{-ix'.p}+\hat{\tilde{a}}_p^{\dagger}e^{ix'.p})(\hat{a}_{p'}^{\dagger}e^{ix.p'}+\hat{\tilde{a}}_{p'}e^{-ix.p'})|0> \\ &= <0|\int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}e^{-ix'.p+ix.p'}\hat{a}_p\hat{a}_{p'}^{\dagger}|0> \\ &= <0|\int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}e^{-ix'.p+ix.p'}(\hat{a}_{p'}^{\dagger}\hat{a}_p+\delta^{(3)}(\vec{p}-\vec{p'})\,\mathbb{1})|0> \\ &= \int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}e^{-ix'.p+ix.p'}\delta^{(3)}(\vec{p}-\vec{p'}) \end{aligned} \tag{1}$$

where we have used respectively the Equations (26), (27), ([25]), (17) and (21)-(31) [recalling the semi-compact notation $|0>\otimes|\tilde{0}>\equiv|0>$ (compact one)] of the course. On the other side,

$$\begin{aligned} iG(x^{\mu}-x'^{\mu}) &= <0|\tau[\hat{\phi}(x^{\nu})\hat{\phi}^{\dagger}(x'^{\nu})]|0>=<0|\hat{\phi}^{\dagger}(x'^{\nu})\hat{\phi}(x^{\nu})|0> \\ &= <0|\int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}(\hat{a}_p^{\dagger}e^{ix'.p}+\hat{\tilde{a}}_p e^{-ix'.p})(\hat{a}_{p'}e^{-ix.p'}+\hat{\tilde{a}}_{p'}^{\dagger}e^{ix.p'})|0> \\ &= <0|\int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}e^{-ix'.p+ix.p'}\hat{\tilde{a}}_p\hat{\tilde{a}}_{p'}^{\dagger}|0> \\ &= <0|\int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}e^{-ix'.p+ix.p'}(\hat{\tilde{a}}_{p'}^{\dagger}\hat{\tilde{a}}_p+\delta^{(3)}(\vec{p}-\vec{p'})\,\mathbb{1})|0> \\ &= \int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}e^{-ix'.p+ix.p'}\delta^{(3)}(\vec{p}-\vec{p'}) \end{aligned} \tag{2}$$

where we have used respectively the Equations (26), (17), ([25]), (27) and (21)-(31) of the course. The comparison of the identical results in Eq.(1) and Eq.(2) allows to conclude that, indeed, one finds:

$$iG(x'^{\mu}-x^{\mu}) = iG(x^{\mu}-x'^{\mu})\,.$$

Notice that for example Eq.(2) leads to,

$$<0|\tau[\hat{\phi}(x^{\nu})\hat{\phi}^{\dagger}(x'^{\nu})]|0>=\int\frac{d^3p}{(2\pi)^3 2E_p}\,e^{i(-x'+x).p} = iG(x^{\mu}-x'^{\mu})$$

(where the last step is based on Eq.([30]) of the course) which allows one to demonstrate the Eq.([32]) of the course.

3. **Evolution operator.-** Check that the evolution operator $\hat{U}_0(t) = e^{-i\hat{H}_0 t}$ ($\hat{H}_0$ being the time-independent free Hamiltonian) for the free quantum state is solution of the equation,

$$i\frac{d}{dt}\hat{U}_0(t) = \hat{H}_0\hat{U}_0(t)$$

and is a unitary operator.

Inserting the $\hat{U}_0(t)$ exponential expression into this equation, we get,

$$i\frac{d}{dt}[\mathbb{1} - i\hat{H}_0 t + \frac{(-i)^2}{2!}\hat{H}_0^2 t^2 + ...] \overbrace{=}^{?} \hat{H}_0[\mathbb{1} - i\hat{H}_0 t + \frac{(-i)^2}{2!}\hat{H}_0^2 t^2 + ...]$$

$$i[0 - i\hat{H}_0 - \frac{1}{2}2\hat{H}_0^2 t + ...] = \hat{H}_0 - i\hat{H}_0^2 t + ...$$

which is a true relation (at first orders here).

Let us now use the $\hat{U}_0(t)$ exponential expression to check the $\hat{U}_0(t)$ unitarity at first order:

$$\hat{U}_0 \underbrace{\hat{U}_0^\dagger}_{\hat{U}_0^{-1}} = (\mathbb{1} - i\hat{H}_0 t + \mathcal{O}(t^2))(\mathbb{1} + i\hat{H}_0^{(\dagger)} t + \mathcal{O}(t^2))$$

$$= \mathbb{1} + i\hat{H}_0 t - i\hat{H}_0 t + \mathcal{O}(t^2)$$

4. *Wick* **contraction.-** In this exercise we will relate the time-ordering, the normal-ordering and the *Wick* contraction.

   (a) Demonstrate the commutation relation, $[\hat{\phi}(x^\mu), \hat{\phi}(x'^\mu)] = 0$, among field operators involving identical time-components $t = t'$.

   (b) Demonstrate the equality property, $:\hat{\phi}(x^\mu)\hat{\phi}(x'^\mu): = :\hat{\phi}(x'^\mu)\hat{\phi}(x^\mu):$, where $:\hat{\phi}(x^\mu)\hat{\phi}(x'^\mu):$ denotes the normal-ordering.

   (c) Using previous question, show that the time-ordering between two fields is given by

$$\tau[\hat{\phi}_1(x^\mu)\hat{\phi}_2(x'^\mu)] = :\hat{\phi}_1(x^\mu)\hat{\phi}_2(x'^\mu): + \underbrace{\hat{\phi}_1(x^\mu)\hat{\phi}_2}(x'^\mu)\,,$$

   where $\underbrace{\hat{\phi}_1(x^\mu)\hat{\phi}_2}(x'^\mu)$ denotes the *Wick* contraction.

***