QUANTUM FIELD THEORY

Tutorials (n'2)

1. **Real field.-** Show that the operator property $\hat{A}^\dagger(p^\mu) = \hat{A}(-p^\mu)$ allows to define a real scalar field accordingly to $\hat{\phi}^\dagger(x^\mu) = \hat{\phi}(x^\mu)$, where [$m$, x^μ and p^μ being respectively the mass parameter, the 4-coordinates and 4-momentum]

$$\hat{\phi}(x^\mu) = \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta(p^\mu p_\mu - m^2) \hat{A}(p^\alpha) e^{-ip^\mu x_\mu} \ .$$

$$\begin{aligned}
\int d^4p\, \delta(p^{\underline{2}} - m^2) \hat{A}(p^\rho) e^{-ip^\mu x_\mu} &= \int d^4p\, \delta(p^{\underline{2}} - m^2) \hat{A}^\dagger(p^\rho) e^{ip^\mu x_\mu} \\
&= \underbrace{\int\int\int\int_{-\infty}^{+\infty}}_{(-1)^4} d^4(-p)\ \delta((-p)^{\underline{2}} - m^2)\ \underbrace{\hat{A}^\dagger(p^\rho)}_{\substack{= \hat{A}(-p^\rho) \\ \text{"if true then equality OK"}}}\ e^{-i(-p^\mu x_\mu)}
\end{aligned}$$

We have used the change of variable, $p'_\mu = -p_\mu$, and renamed the variable, $p'_\mu \to p_\mu$, as it is an integration variable.

2. **Canonical commutation relations.-** Show that the following operator properties,

$$[\hat{a}_p, \hat{a}^\dagger_{p'}] = \delta^{(3)}(\vec{p} - \vec{p}\,'), \ [\hat{a}_p, \hat{a}_{p'}] = 0, \ [\hat{a}^\dagger_p, \hat{a}^\dagger_{p'}] = 0, \tag{1}$$

allow to recover the commutation relation between the real scalar field and its conjugate momentum, namely $[\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$, where [$x^\mu$ and p^μ being respectively the 4-coordinates and 4-momentum while $E_p = \sqrt{\vec{p}^{\,2} + m^2}$]

$$\hat{\phi}(x^\mu) = \int d^3p \frac{1}{\sqrt{(2\pi)^3 2E_p}} \left(\hat{a}_p\, e^{-ip^\mu x_\mu} + \hat{a}^\dagger_p\, e^{ip^\mu x_\mu}\right) . \tag{2}$$

$$\begin{aligned}
[\hat{\phi}(t, \vec{x}), \hat{\pi}_{(\phi)}(t, \vec{y})] = \int \frac{\overbrace{d^3p}^{\substack{\text{4-momentum} \\ p^\mu \text{ with } p^0 = E_p}}}{\sqrt{(2\pi)^3 2E_p}} \int \frac{d^3p'\ i\sqrt{E_{p'}}}{\sqrt{(2\pi)^3}\sqrt{2}} \Bigg(&\overbrace{[\hat{a}_p e^{-ix.p}, -\hat{a}_{p'} e^{-ip'.y}]}^{=0 \text{ from Eq.([18]) of course}} + \overbrace{[\hat{a}_p e^{-ix.p}, \hat{a}^\dagger_{p'} e^{ip'.y}]}^{\ni \delta^{(3)}(\vec{p} - \vec{p}\,') e^{-iE_p t + iE_{p'} t}} \\
&+ \overbrace{[\hat{a}^\dagger_p e^{ix.p}, -\hat{a}_{p'} e^{-ip'.y}]}^{\ni \delta^{(3)}(\vec{p} - \vec{p}\,') e^{iE_p t - iE_{p'} t}} + \overbrace{[\hat{a}^\dagger_p e^{ix.p}, \hat{a}^\dagger_{p'} e^{ip'.y}]}^{=0 \text{ from Eq.([18])}} \Bigg)
\end{aligned}$$

$$(\star) = \int_{-\infty}^{+\infty} \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \frac{i\sqrt{E_p}}{\sqrt{2}} \left[e^{i\vec{p}.(\vec{x}-\vec{y})} e^{-iE_pt+iE_pt} + e^{-i\vec{p}.(\vec{x}-\vec{y})} e^{iE_pt-iE_pt} \right]$$
$$= i\,\frac{1}{2}\frac{1}{(2\pi)^3}\left[(2\pi)^3\delta^{(3)}(\vec{x}-\vec{y}) + (2\pi)^3\delta^{(3)}(\vec{y}-\vec{x})\right]$$
$$= i\,\delta^{(3)}(\vec{x}-\vec{y})$$

where we have used the equality, Fourier-Tr(1_p) = $2\pi\delta(x)$, without any conventional constant factor in the Fourier transform.

Then deduce the relation $[\hat{\phi}(t,\vec{x}), \partial_i\hat{\phi}(t,\vec{y})] = 0$ where $\partial_i = \frac{\partial}{\partial x^i}$ $[i = 1, 2, 3]$ denotes the spatial derivative.

$$\partial_i\hat{\phi}(t,\vec{y}) \;=\; \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}(ip_i)(-\hat{a}_p e^{-iy.p} + \hat{a}_p^\dagger e^{iy.p})$$

$$[\hat{\phi}, \partial_i\hat{\phi}] = ...\,(\star) = \int_{-\infty}^{+\infty} \frac{d^3p}{(2\pi)^3}\frac{ip_i}{2E_p} e^{i\vec{p}.(\vec{x}-\vec{y})} + \underbrace{\int\int\int_{+\infty}^{-\infty}}_{\to(-1)\text{ from } \vec{p}'=-\vec{p}} \frac{d^3(-p)}{(2\pi)^3}\frac{i(-p_i)}{2\,\underbrace{E_p}_{=E_{-\vec{p}}\equiv(\vec{p}^2+m^2)^{1/2}}} e^{-i\vec{p}.(\vec{x}-\vec{y})} = 0$$

where we have used: $x.p = p_0x^0 + p_jx^j$ and made the replacement $\sqrt{E_{p^{(\prime)}}} \to \frac{p_i^{(\prime)}}{\sqrt{E_{p^{(\prime)}}}}$ in the intermediate result $(\star)$ of previous question as the respective starting points, $[\hat{\phi}(t,\vec{x}), \hat{\pi}_{(\phi)}(t,\vec{y})]$ and $[\hat{\phi}(t,\vec{x}), \partial_i\hat{\phi}(t,\vec{y})] = 0$, are identical precisely up to this replacement $\hat{\pi}_{(\phi)}(t,\vec{y}) \to \partial_i\hat{\phi}(t,\vec{y})$.

3. **Hamiltonian expression.-** Calculate the Hamiltonian

$$\hat{H} = \int d^3x\,\hat{\mathcal{H}} = \int d^3x\,\left\{\frac{1}{2}(\partial_0\hat{\phi})^2 + \frac{1}{2}\vec{\nabla}\hat{\phi}.\vec{\nabla}\hat{\phi} + \frac{1}{2}m^2\hat{\phi}^2\right\}$$

where $\partial_0 = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}$ represents the time derivative [within the natural unit system where $c = 1$]. For this purpose, make use of Equation (2) as well as the three-*Fourier* transformation formula $\int d^3x\, e^{i\vec{p}.\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{p})$. Express the obtained result in terms of E_p, $\hat{a}_p$ and $\hat{a}_p^\dagger$ exclusively.

• The first term of $\hat{H}$ is proportional to,

$$\int d^3x\,(\partial_o\phi)^2 \underbrace{=}_{\text{Eq.}[11],[17]} \int d^3x\,(-1)\int\frac{d^3p}{\sqrt{2(2\pi)^3}}E_p^{1/2}\int\frac{d^3p'}{\sqrt{2(2\pi)^3}}E_{p'}^{1/2}$$
$$(-\hat{a}_p e^{-ix.p} + \hat{a}_p^\dagger e^{ix.p})(-\hat{a}_{p'} e^{-ip'.x} + \hat{a}_{p'}^\dagger e^{ip'.x})$$
$$= \int d^3x\,(-1)\int\frac{d^3p}{\sqrt{2(2\pi)^3}}E_p^{1/2}\int\frac{d^3p'}{\sqrt{2(2\pi)^3}}E_{p'}^{1/2}$$
$$\left(\hat{a}_p\hat{a}_{p'}\underbrace{e^{-ix.(p+p')}}_{\ni e^{-i(E_p+E_{p'})t}} - \hat{a}_p\hat{a}_{p'}^\dagger\underbrace{e^{ix.(p'-p)}}_{\ni e^{i(E_{p'}-E_p)t}} - \hat{a}_p^\dagger\hat{a}_{p'}\underbrace{e^{ix.(p-p')}}_{\ni e^{i(E_p-E_{p'})t}} + \hat{a}_p^\dagger\hat{a}_{p'}^\dagger\underbrace{e^{ix.(p+p')}}_{\ni e^{i(E_p+E_{p'})t}}\right)$$

and let us use now, $\int \frac{d^3x}{(2\pi)^3}\, e^{i\vec{p}.\vec{x}} = \delta^{(3)}(\vec{p})$:

$$\int d^3x (\partial_o\hat{\phi})^2 = -\int d^3p\sqrt{\frac{E_p}{2}}\int d^3p'\sqrt{\frac{E_{p'}}{2}}$$
$$\Big(\hat{a}_p\hat{a}_{p'}\, e^{-i(E_p+E_{p'})t}\,\delta^{(3)}(\vec{p}+\vec{p'}) - \hat{a}_p\hat{a}^\dagger_{p'}\, e^{i(E_{p'}-E_p)t}\,\delta^{(3)}(\vec{p}-\vec{p'})$$
$$- \hat{a}^\dagger_p\hat{a}_{p'}\, e^{i(E_p-E_{p'})t}\,\delta^{(3)}(-\vec{p}+\vec{p'}) + \hat{a}^\dagger_p\hat{a}^\dagger_{p'}\, e^{i(E_p+E_{p'})t}\,\delta^{(3)}(-\vec{p}-\vec{p'})\Big)$$
$$= -\int d^3p\, \frac{E_p}{2}\Big(\overbrace{\hat{a}(\vec{p})\hat{a}(-\vec{p})}^{(\star)}\, e^{-i2E_pt} - \hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) - \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \hat{a}^\dagger(\vec{p})\hat{a}^\dagger(-\vec{p})e^{2iE_pt}\Big).$$

$$(\star)\begin{cases} \vec{p'} = \pm\vec{p} \text{ and } a(p^\mu) = a(E_p,\vec{p}) \underbrace{\equiv}_{E_p = E_p(\pm\vec{p})} a(\vec{p}) \\ E_{p'} = E_p \end{cases}$$

• The second term of $\hat{H}$ is proportional to,

$$\int d^3x\, \vec{\nabla}\hat{\phi}.\vec{\nabla}\hat{\phi} = \int d^3x \int \frac{d^3p\; ip_i}{\sqrt{(2\pi)^3 2E_p}}\int \frac{d^3p'\; ip'_i}{\sqrt{(2\pi)^3 2E_{p'}}}(-\hat{a}_p e^{-ix.p}+\hat{a}^\dagger_p e^{ix.p})(-\hat{a}_{p'}e^{-ip'.x}+\hat{a}^\dagger_{p'}e^{ip'.x})$$

$$\left(\text{since Eq.}[16] : -ix.p \ni ix^jp^j\ ,\ \ \partial_i\phi = \int \frac{d^3p\; ip^i}{\sqrt{(2\pi)^3 2E_p}}(\hat{a}_p e^{-ix.p} - \hat{a}^\dagger_p e^{ix.p})\right)$$

$$= -\int d^3p\, \frac{1}{2}\frac{p^ip^i}{E_p}\Big(-\hat{a}(\vec{p})\hat{a}(-\vec{p})e^{-2iE_pt} - \hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) - \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) - \hat{a}^\dagger(\vec{p})\hat{a}^\dagger(-\vec{p})e^{2iE_pt}\Big)\ .$$

Difference with starting point of previous calculation: $\sqrt{E_{p^{(\prime)}}} \to \frac{p_i^{(\prime)}}{E^{1/2}_{p^{(\prime)}}}$.

• The third term of $\hat{H}$ is proportional to,

$$\int d^3x\, m^2\hat{\phi}^2 \underbrace{=}_{\text{Eq.}[16]} \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int \frac{d^3p\,'}{\sqrt{(2\pi)^3 2E_{p'}}}\, m^2\int d^3x\,(\hat{a}_p e^{-ix.p} + \hat{a}^\dagger_p e^{ix.p})(\hat{a}_{p'}e^{-ip'.x} + \hat{a}^\dagger_{p'}e^{ip'.x})$$
$$= m^2\int d^3p\,\frac{1}{2E_p}(\hat{a}(\vec{p})\hat{a}(-\vec{p})e^{-2iE_pt} + \hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) + \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \hat{a}^\dagger(\vec{p})\hat{a}^\dagger(-\vec{p})e^{2iE_pt})\ .$$

Differences with starting point of previous calculation:
$\sqrt{E_{p^{(\prime)}}} \to E^{-1/2}_{p^{(\prime)}}$, $1 \to -m^2$, $\hat{a}(\vec{p^{(\prime)}}) \to -\hat{a}(\vec{p^{(\prime)}})$.

$$\Rightarrow \int d^3x((\vec{\nabla}\hat{\phi})^2 + m^2\hat{\phi}^2)$$
$$= \int d^3p\,\frac{1}{2E_p}\underbrace{[\vec{p}^2 + m^2]}_{E_p^2}\Big(\hat{a}(\vec{p})\hat{a}(-\vec{p})e^{-2iE_pt} + \hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) + \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \hat{a}^\dagger(\vec{p})\hat{a}^\dagger(-\vec{p})e^{2iE_pt}\Big)$$

$$\Rightarrow \quad 2\hat{H} = \int d^3p\, \frac{E_p}{2}\left(2\,\hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) + 2\,\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})\right).$$

4. **Hamiltonian commutator relations.-** By using Equation (1), demonstrate that the two following relations on commutators, involving the Hamiltonian operator, are correct.

   (a) $[\hat{H}, \hat{a}_{p'}] = -E_{p'}\,\hat{a}_{p'}$.

   (b) $[\hat{H}, \hat{a}^\dagger_{p'}] = E_{p'}\,\hat{a}^\dagger_{p'}$.

(a)

$$\begin{aligned}
[\hat{H}, \hat{a}(\vec{p'})] \underbrace{=}_{\text{Eq.[19]}} & \frac{1}{2}\int d^3p\, E_p\Big[\hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) + \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}), \hat{a}(\vec{p'})\Big] \\
&= \frac{1}{2}\int d^3p\, E_p\Big[\hat{a}(\vec{p})\underbrace{[\hat{a}^\dagger(\vec{p}), \hat{a}(\vec{p'})]}_{-\delta^{(3)}(\vec{p}-\vec{p'})} + \underbrace{[\hat{a}(\vec{p}), \hat{a}(\vec{p'})]}_{=0\ \text{(Eq.[18])}}\hat{a}^\dagger(\vec{p}) \\
&+ \hat{a}^\dagger(\vec{p})\underbrace{[\hat{a}(\vec{p}), \hat{a}(\vec{p'})]}_{=0\ \text{(Eq.[18])}} + \underbrace{[\hat{a}^\dagger(\vec{p}), \hat{a}(\vec{p'})]}_{-\delta^{(3)}(\vec{p}-\vec{p'})}\hat{a}(\vec{p})\Big] \\
&= -\frac{1}{2}(E_{p'}\hat{a}_{(\vec{p'})} + E_{p'}\hat{a}_{(\vec{p'})}) = -E_{p'}\,\hat{a}(\vec{p'})
\end{aligned}$$

(b)

$$\begin{aligned}
[\hat{H}, \hat{a}^\dagger(\vec{p'})] &= \frac{1}{2}\int d^3p\, E_p\Big[[\hat{a}_p\hat{a}^\dagger_p, \hat{a}^\dagger_{p'}] + [\hat{a}^\dagger_p\hat{a}_p, \hat{a}^\dagger_{p'}]\Big] \\
&= \frac{1}{2}\int d^3p\, E_p\Big[\hat{a}(\vec{p})\underbrace{[\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{p'})]}_{=0\ \text{(Eq.[18])}} + \underbrace{[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p'})]}_{\delta^{(3)}(\vec{p}-\vec{p'})}\hat{a}^\dagger(\vec{p}) \\
&+ \hat{a}^\dagger(\vec{p})\underbrace{[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p'})]}_{\delta^{(3)}(\vec{p}-\vec{p'})} + \underbrace{[\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{p'})]}_{=0\ \text{(Eq.[18])}}\hat{a}(\vec{p})\Big] \\
&= \frac{1}{2}(E_{p'}\hat{a}^\dagger(\vec{p'}) + \hat{a}^\dagger(\vec{p'})E_{p'}) \\
&= E_{p'}\,\hat{a}^\dagger(\vec{p'})
\end{aligned}$$

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