Version 2.4

Quantum Field Theory

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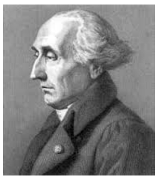
Abstract

The present notes provide exclusively a synthesis of the formalism presented in the lecture series “Quantum Field Theory” (part of the Major Course “Statistical and Quantum Mechanics”) at the 1st year Master *General Physics* of the Paris-Saclay University. In these lectures, I provide a progressive introduction to the so-called second quantisation, for the simplest case of a scalar field (spinless). All the non-trivial calculations, including their intermediate steps, are performed explicitly. There are several ways to treat the second quantisation: the canonical quantisation is presented in these lectures. The final goal is to be able to calculate the transition amplitude for the simplest $1 \to 2$-body particle decay, at the level of one loop with respect to quantum corrections.

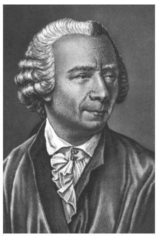

G.-C.Wick


J.-L.Lagrange


W.Gordon


E.Schrödinger


V.A.Fock


L.Euler


W.R.Hamilton


O.Klein


R.Feynman


E.Noether

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Motivations

Quantum Mechanics must be extended to (special) relativity framework:

→ consider relativistic fundamental equations

→ be able to write a covariant formalism

→ go beyond the untenable potential description [implying the transfer of information at an infinite speed].

"like: $E^2 = \vec{p}^2c^2 + m^2c^4$"

"$V(x,t)$ non relativistic $\rightarrow V(\phi(x^\mu))$ non relat. via $\phi(x)$"

The Quantum Field Theory (QFT), introducing the creation/annihilation operators in a Fock space (second quantisation), is more formal and complete than the sole perturbation theory applied to relativistic quantum mechanics [PNU Major]. It

- allows a systematic reaction amplitude calculation for any Feynman diagram
- includes the possibility of absorption/creation of any elementary particle
- represents a 'natural' extension from the point of view of the universal Analytical Mechanics (Lagrangian and Hamiltonian approaches):

"including quantum corrections at the loop level"

Overview of the fundamental variables and equations

Classical Physics $[x, \dot{x}/p, t]$
EOM: 2nd Newton's law
$(-V'(x) = m\,\ddot{x})$
$L(x, \dot{x}, t) = T - V$
$\rightarrow \{p(t), x(t)\}$, $H(x, \dot{x}, t)$
→ EOM
H (real) ≡ Energy values.

1st ⇒

Quantum Mechanics $[\hat{X}, \hat{P}, t]$
EOM: Newton's law ↔ Schrödinger equation
(EOM $(\hat{X}, \hat{P}, t) \leftrightarrow \hat{H}\phi = i\hbar\,\partial_t\phi$, $\hat{H}$ in $\mathbb{H}$)
$\hat{L}(\hat{X}, \hat{P}, t) = \hat{T} - \hat{V}$
$[\hat{P}(t), \hat{X}(t)]$, $\hat{H}(\hat{X}, \hat{P}, t)$
EOM
Real $\hat{H}$ eigenvalues $(\hat{H}^\dagger = \hat{H})$ ≡ Energy values.

'new variables' ↙

Relativistic Quantum Mechanics $[\phi/\psi/A^\mu, \partial_\mu\phi, x^\nu]$
EOM: Klein-Gordon/Dirac equat.
($(\partial_\mu\partial^\mu + m^2)\phi = 0$, $\phi(x^\mu)$: field)
$\mathcal{L}(\phi, \partial_\mu\phi)$
$\rightarrow \{\pi(x^\nu), \phi(x^\nu)\}$, $\mathcal{H}(\phi, \partial_\mu\phi)$
→ EOM
$\mathcal{H}$ (real) ≡ Energy densities $[f(\phi)]$.

2nd ⇒

Quantum Field Theory $[\hat{\phi}, \partial_\mu\hat{\phi}, x^\nu]$
EOM: Klein-Gordon equation
($(\partial_\mu\partial^\mu + m^2)\hat{\phi} = 0$, $\hat{H}\Psi = i\hbar\,\partial_t\Psi$, $\hat{H}$ in $\mathbb{F}$)
$\hat{\mathcal{L}}(\hat{\phi}, \partial_\mu\hat{\phi})$
$[\hat{\pi}(x^\nu), \hat{\phi}(x^\nu)]$, $\hat{\mathcal{H}}(\hat{\phi}, \partial_\mu\hat{\phi})$
EOM
Real $\hat{\mathcal{H}}$ eigenvalues $(\hat{\mathcal{H}}^\dagger = \hat{\mathcal{H}})$ ≡ En. densities.

1 Fundamental physics overview from analytical mechanics

1.1 Classical physics

The standard context of rigid body coordinates covers point-like systems (localisable at a given position, say x or q).

A non-relativistic and classical (non-quantum) system is described by variables (possibly spatial coordinates) noted here $q_r(t)$ $[r = 1, 2, \dots]$ and time derivatives (could be velocities) $\dot{q}_r(t)$.

1.1.1 Principle of least action

The Lagrangian (L) mechanics introduces the so-called action (defined mathematically as a functional):

$$\mathcal{A}[q] = \int_{t_1}^{t_2} \mathrm{d}t \; L\left(q_r(t), \dot{q}_r(t), t\right) . \tag{1}$$

Least action principle: Among all trajectories ($\delta q_r(t)$ infinitesimal variations in the (q_r, t) space) that join $q_r(t_1)$ to $q_r(t_2)$ ($\delta q_r(t_1) = \delta q_r(t_2) = 0$ to define the path boundaries), the system follows the one for which $\mathcal{A}$ is stationary ($\delta\mathcal{A} = 0$; minimum).

This principle leads to the Euler-Lagrange equations i.e. the Equations Of Motion (EOM), reading as

$$\boxed{\frac{d}{dt}\left(\frac{\partial L(q_r(t), \dot{q}_r(t), t)}{\partial \dot{q}_r(t)}\right) = \frac{\partial L(q_r(t), \dot{q}_r(t), t)}{\partial q_r(t)}} \tag{2}$$

Standard (partial) derivatives (∂) are used as in this result the same variables arise in the numerators and denominators (and functions depending on the same time variable must hold in the left and right-hand sides of the equality).

1.1.2 The conjugate momentum and Hamiltonian

The momentum 'conjugate' to q_r is defined by,

$$p_r(t) \;\widehat{=}\; \frac{\partial L(q_r, \dot{q}_r, t)}{\partial \dot{q}_r(t)} . \tag{3}$$

The same time variable appears in the left and right-hand sides of this equation and the standard derivative is involved for the same variable $\dot{q}_r(t)$ (as same t variable) in the numerator and denominator.

Hamiltonian definition (represents the system energy as checked in Tutorials 1, Exercise 1):

$$H(q_r(t), p_r(t)) \;\widehat{=}\; p_r(t)\; \underbrace{\dot{q}_r(q_r, p_r) - L(q_r(t), \dot{q}_r(q_r, p_r), t)}_{\text{'Eq.(3) is generally invertible for } \dot{q}_r\text{'}} . \tag{4}$$

Notice that $p_r\dot{q}_r$ uses the Einstein's convention of implicit summation over the repeated index r, $\sum_r p_r\dot{q}_r$. Again the same t variable is involved in each part of this equality. H is real, like L.

By differentiation [in case of $L(V(q(t)))$ in contrast with a direct time dependence $L(V(t))$],

$$\mathrm{d}H = \dot{q}_r \mathrm{d}p_r + p_r \mathrm{d}\dot{q}_r - \Big(\underbrace{\frac{\partial L}{\partial q_r}}_{= \dot{p}_r \text{ via Eq.(2) and } \frac{d}{dt}\text{Eq.(3)}} \mathrm{d}q_r + \frac{\partial L}{\partial \dot{q}_r}\mathrm{d}\dot{q}_r \Big)$$

$$\Leftrightarrow \quad \frac{\partial H}{\partial p_r} = \dot{q}_r, \quad \frac{\partial H}{\partial q_r} = -\dot{p}_r \text{ (after a cancellation).} \tag{5}$$

Those two equations are the Hamilton-Jacobi equations and are equivalent to Eq.(2) (EOM) and Eq.(3): this approach is the Hamiltonian mechanics, the alternative analytical branch.

1.1.3 Poisson bracket

Within a given physical context, some quantities can turn out to be conserved – a consideration which takes the following form,

$$\frac{d}{dt} f(q_r(t), p_r(t)) \; = \; 0\,,$$

where $f(q_r(t), p_r(t))$ is a function of the dynamical variables. This general remark constitutes a motivation to write down the total time derivation of a generic function, in terms of the partial derivatives:

$$\frac{d}{dt} f(q_r(t), p_r(t)) = \frac{\partial f}{\partial t} + \sum_r \left[\frac{\partial f}{\partial q_r}\frac{\partial q_r}{\partial t} + \frac{\partial f}{\partial p_r}\frac{\partial p_r}{\partial t} \right] ,$$

the first term of this expression indicating the possible direct time dependence of the quantity f. Applying the Hamilton-Jacobi equations (5), one obtains the equality,

$$\frac{d}{dt} f(q_r(t), p_r(t)) = \frac{\partial f}{\partial t} + \sum_r \left[\frac{\partial f}{\partial q_r}\frac{\partial H}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial H}{\partial q_r} \right] ,$$

that can be rewritten as,

$$\frac{df}{dt} \; = \; \frac{\partial f}{\partial t} \; + \; [f, H]_P\,, \tag{6}$$

by introducing the Poisson bracket definition for any pair of functions of the dynamical variables:

$$[f_1, f_2]_P \equiv \{f_1(t), f_2(t)\} \,\widehat{=}\, \frac{\partial f_1(t)}{\partial q_r(t)}\frac{\partial f_2(t)}{\partial p_r(t)} - \frac{\partial f_1(t)}{\partial p_{r'}(t)}\frac{\partial f_2(t)}{\partial q_{r'}(t)}\;. \tag{7}$$

Once more, same time variable in each term (by construction) and hence standard derivatives. Einstein convention still used on the indices r and r' [without co/contra-variance position (down/up) notions]. The dimension analysis of Eq.(6)-(7) can be checked e.g. via Tutorials 1, Exercise 1 (where q is a spatial coordinate and p a physical momentum):

$$\frac{[f]}{T} = \frac{[f]\left(M\frac{L^2}{T^2}\right)}{L.M\frac{L}{T}} \ .$$

Analytical formalism generalisable to special relativity: $\begin{cases} \text{4-force: } f^\mu = \frac{dp^\mu}{d\tau} \\ \text{4-velocity: } v^\mu = p^\mu/m \end{cases}$

where τ denotes the time in rest frame and $v^\mu = (\gamma c, \gamma\vec{v})$.

1.2 Quantum mechanics

Let us describe the path for the extension from classical physics to quantum mechanics (to be applied in Section 1.4), also called first quantisation.

Eq.(1) $\mapsto$ e.g. take $\hat{L}(\hat{X},\hat{P},t)$ with $\hat{P}_x\phi(x,t) = -i\hbar\partial_x\phi(x,t)$

Eq.(2) $\mapsto$ e.g. write directly the EOM$(\hat{X},\hat{P},t)$ (no derivatives w.r.t. operators) or the fundamental Schrödinger's equation: $\hat{H}\phi = i\hbar\partial_t\phi(\vec{x},t)$

Eq.(3) $\mapsto$ extend directly to e.g. $\hat{\vec{P}}$ (same comment on the derivative)

Eq.(4) $\mapsto$ extend directly to e.g. $\hat{H} = \frac{\hat{\vec{P}}^2}{2m} + \hat{V}(\hat{\vec{X}})$ (same comment on the derivative)

Eq.(7): $\{\ldots,\ldots\} \mapsto [\ldots,\ldots]/i\hbar$ with e.g. $[x_r,p_s]_P = \delta_{rs} \mapsto [\hat{X}_r,\hat{P}_s] = i\hbar\delta_{rs}$ (from Eq.(1) of Tutorials 1, Exercise 2, Question a) which is induced by $\hat{P}$ expression and is nothing else but the operator form of the Heisenberg principle of uncertainty [indices r,s for spatial dimensions] also comparable to $[\hat{H},\hat{O}] = i\hbar\partial_t\hat{O}$

Eq.(6) $\mapsto$ Ehrenfest theorem (having in mind that $f|_{\text{classical}} =<\phi(t)|\hat{F}|\phi(t)>$ like e.g. $x|_{\text{classical}} =<\phi(t)|\hat{X}|\phi(t)>$) that can be demonstrated using the Schrödinger equation:

$$\frac{d}{dt} <\phi(t)|\hat{F}|\phi(t)> = <\frac{\partial\hat{F}}{\partial t}>_{|\phi(t)>} + \frac{1}{i\hbar} <[\hat{F},\hat{H}]>_{|\phi(t)>} \quad \text{with} \quad <\vec{x}|\phi(t)>=\phi(\vec{x},t) \ . \tag{8}$$

1.3 Relativistic quantum mechanics: field theory

1.3.1 Principle of least action

$\begin{cases} \text{Relavistic extension } (x^\nu,\ldots) \\ \text{Quantum mechanics } (\phi,\ldots) \end{cases} \Rightarrow \text{introduce fields } \phi(x^\nu)$

Action defined now as,

$$\mathcal{A} = \frac{1}{c}\int \underbrace{d^4x}_{\text{'treat space/time on same footing'}}\ \mathcal{L}\Big(\phi^A(x^\nu), \partial_\mu\phi^A(x^\nu)\Big) = \int \frac{dx^0}{c}\left(\int d^3x\,\mathcal{L}\right) = \int dt\,(L). \tag{9}$$

Remarks:
- same x^ν in each term by definition like in Eq.(1) with the time variable
- separate Lorentz invariances of the pieces $\frac{1}{c}$, $\int d^4x$ and $\mathcal{L}$
- $V(x^\nu)$ non relativistic $\to V(\phi(x^\nu))$ non relativistic via $\phi(x^\nu)$
- still $[\mathcal{A}] = [\hbar]$ as will be seen
- ϕ^A is a function of x^ν (the label A can indicate field species, spin components, gauge group indices,...)
- $\begin{cases} \mathcal{L} \text{ is a function of } \phi^A \\ \mathcal{A} \text{ is a functional of } \phi^A \end{cases}$

Least action principle: Among all trajectories ($\delta\phi^A(x^\nu)$ infinitesimal variations in the (ϕ^A, x^ν) space) that join $\phi^A(x_1^\nu)$ to $\phi^A(x_2^\nu)$ ($\delta\phi^A(x_1^\nu) = \delta\phi^A(x_2^\nu) = 0$ to define the path boundaries), the system follows the one for which $\mathcal{A}$ is stationary ($\delta\mathcal{A} = 0$; minimum).

Induced Euler-Lagrange equations:

$$\boxed{\partial_\mu \frac{\partial\mathcal{L}(\phi^A(x^\nu), \partial_\mu\phi^A(x^\nu))}{\partial(\partial_\mu\phi^A(x^\nu))} = \frac{\partial\mathcal{L}(\phi^A(x^\nu), \partial_\mu\phi^A(x^\nu))}{\partial\phi^A(x^\nu)}} \quad (\forall A) \tag{10}$$

The same 4-coordinates appear in each term of this equation as in Eq.(2) for the time variable (same reasons). Hence the standard derivative is involved.

1.3.2 The conjugate momentum and Hamiltonian

Conjugate momentum definition:

$$\pi_A(x^\nu) \hat{=} \frac{\partial\mathcal{L}(\phi^A(x^\nu), \partial_\mu\phi^A(x^\nu))}{\partial\dot{\phi}^A(x^\nu)}\ . \tag{11}$$

The same 4-coordinates appear in each term of this equation as in Eq.(3) so that the standard derivative is involved.

Hamilton density definition [dimensions correct given those of Eq.(4) yet checked]:

$$\mathcal{H}\Big(\phi^A(x^\nu), \pi_A(x^\nu), \vec{\nabla}\phi^A(x^\nu)\Big) \hat{=} \pi_A(x^\nu)\dot{\phi}^A(x^\nu) - \mathcal{L}\Big(\phi^A(x^\nu), \partial_\mu\phi^A(\pi_A, \phi^A, \vec{\nabla}\phi^A)\Big)\ . \tag{12}$$

The same 4-coordinates are involved in each term of this equation similarly to Eq.(4) and the dependence $\partial_\mu\phi^A(\pi_A, \phi^A, \vec{\nabla}\phi^A)$ is a general consequence of the inversion of Eq.(11). $\mathcal{H}$ is real like $\mathcal{L}$ since it is defined through $H = \int d^3x\,\mathcal{H}$. Note that $\vec{\nabla}\phi^A$ is the new part here from $\partial_\mu\phi^A$.

1.3.3 Poisson bracket

Poisson bracket generalisation from Eq.(7):

$$
\begin{aligned}
[\phi^A(t,\vec{x}),\pi_B(t,\vec{y})]_P &\hat{=} \int d^3z\,\left(\frac{\delta\phi^A(t,\vec{x})}{\delta\phi_{C'}(t,\vec{z})}\frac{\delta\pi_B(t,\vec{y})}{\delta\pi_{C'}(t,\vec{z})}-\frac{\delta\phi^A(t,\vec{x})}{\delta\pi_C(t,\vec{z})}\frac{\delta\pi_B(t,\vec{y})}{\delta\phi_C(t,\vec{z})}\right)\\
&= \int d^3z\;\delta^A_{C'}\,\delta^{(3)}(\vec{x}-\vec{z})\;\delta^{C'}_B\,\delta^{(3)}(\vec{y}-\vec{z})\\
&= \delta^A_B\,\delta^{(3)}(\vec{x}-\vec{y})
\end{aligned}
\tag{[6]}
$$

where different coordinates $\vec{x}$ and $\vec{y}$ are considered to be generic. There is a continuous summation on the variable z, as the discrete index C is summed, so that no new dependence arises on the right-hand side of the equation. Like in Eq.(7), same time variable in each term so that the following derivatives must be introduced. One can show through the introduction of functional derivatives that,

$$
\frac{\delta\phi^B(t,\vec{y})}{\delta\phi^A(t,\vec{x})}=\delta^B_A\delta^{(3)}(\vec{x}-\vec{y})\ ,\quad \frac{\delta\dot{\phi}^B(t,\vec{y})}{\delta\dot{\phi}^A(t,\vec{x})}=\delta^B_A\delta^{(3)}(\vec{x}-\vec{y})\ . \tag{13}
$$

Notice the non-trivial dimension analysis of this derivation.

Eq.(6) is extended to the equality

$$
\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t}+\ [f,H]_P \tag{14}
$$

with now: $f(x^\alpha)=f(\phi_A(x^\nu),\pi_A(x^\nu))$.

1.4 Quantum field theory

The second quantisation starts from the following procedure, following Section 1.2,

Eq.(9) $\mapsto\ \hat{\mathcal{L}}(\hat{\phi}^A(x^\nu),\partial_\mu\hat{\phi}^A(x^\nu))$

Eq.(10) $\mapsto$ EOM$(\hat{\phi}^A(x^\nu),\partial_\mu\hat{\phi}^A(x^\nu)$ (no derivatives w.r.t. operators)

Eq.(11) $\mapsto$ extend directly to $\hat{\pi}_A(x^\nu)=f\left(\hat{\phi}^A(x^\nu)\right)$ (same comment on the derivative)

Eq.(12) $\mapsto$ extend directly to $\hat{\mathcal{H}}\,\hat{=}\,\hat{\pi}_A(x^\nu)\dot{\hat{\phi}}^A(x^\nu)-\hat{\mathcal{L}}$ (same comment on the derivative), $\hat{\mathcal{L}}$ being hermitian like $\hat{\mathcal{H}}$

Eq.([6]) $\mapsto$:

$$
[\hat{\phi}^A(t,\vec{x}),\hat{\pi}_B(t,\vec{y})]=i\hbar\,\delta^A_B\delta^{(3)}(\vec{x}-\vec{y})\,\hat{1}\,, \tag{[7]}
$$

since $\{\ldots,\ldots\}\mapsto[\ldots,\ldots]/i\hbar$

Eq.(14) $\mapsto$ taking $f|_{\rm classical} =< \eta(t)|\hat{F}|\eta(t) >$ like e.g. $\phi|_{\rm classical} =< \eta(t)|\hat{\phi}|\eta(t) >$ in a new Hilbert space,

$$\frac{d}{dt} < \eta(t)|\hat{F}|\eta(t) > = < \frac{\partial \hat{F}}{\partial t} >_{|\eta(t)>} + \frac{1}{i\hbar} < [\hat{F}, \hat{H}] >_{|\eta(t)>} \quad \text{with} \quad < \vec{x}|\eta(t) >= \eta(\vec{x}, t) \ . \tag{15}$$

2 Quantisation of the real scalar field

We study now explicitly the Sections (1.3) on classical field theory and (1.4) on quantum field theory.

2.1 The Lagrangien and its Hamiltonian

2.1.1 The actions

The lagrangian density for a classical field (wave function in relativistic context)

$$\mathcal{A} = \int d^4x \ \mathcal{L} \ , \quad \mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \ \equiv \ \frac{1}{2}[(\partial_0\phi)^2 - (\vec{\nabla}\phi)^2] - \frac{1}{2}m^2\phi^2$$

describes the kinematics and mass of a spinless particle. Here we consider a real field (no charge): $\phi^* \hat{=} \phi$.

This Lagrangian density is extended for a field operator:

$$\hat{\mathcal{A}} = \int d^4x \ \hat{\mathcal{L}} \ , \quad \boxed{\hat{\mathcal{L}} = \frac{1}{2}\partial^\mu\hat{\phi}\partial_\mu\hat{\phi} - \frac{1}{2}m^2\hat{\phi}^2} \ \equiv \ \frac{1}{2}[(\partial_0\hat{\phi})^2 - (\vec{\nabla}\hat{\phi})^2] - \frac{1}{2}m^2\hat{\phi}^2 \qquad ([8])$$

involving an Hermitian field operator: $\hat{\phi}^\dagger \hat{=} \hat{\phi}$.

2.1.2 Dimensional analysis

In the natural unit system ($\hbar = c = 1$),

$$[\mathcal{A}] = [\hbar] = 1 \Rightarrow [\mathcal{L}] = [E]^4 \Rightarrow [\phi] = [E] \quad ([m] = [E]) \ .$$

2.1.3 Euler-Lagrange equation

$$\text{Eq.(10)} : \frac{\partial\mathcal{L}}{\partial\phi} = \partial_\rho[\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}] \quad - m^2\phi = \partial_\rho[\frac{1}{2}\underbrace{\delta^\rho_\mu\partial^\mu\phi}_{\partial^\rho\phi} + "]$$

Quantum relativistic equation of motion, $(\Box + m^2)\phi = 0$, extended to the field operator:

$$\boxed{(\Box + m^2)\hat{\phi} = 0} \text{ (Klein-Gordon equation)} \qquad ([9])$$

This field operator is (must be) a Lorentz scalar: $\hat{\phi}(x^\mu) = \hat{\phi}'(x'^\mu)$.

2.1.4 Hamiltonian density

Conjugate momentum:

$$\pi_{(\phi)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi \quad \text{[see Section 1.3]}$$

Quantised version:

$$\hat{\pi}_{(\phi)} = \partial_0 \hat{\phi} \quad \text{[see Section 1.4]} \tag{[11]}$$

Hamiltonian density:

$$\mathcal{H} = \pi_{(\phi)}\dot{\phi} - \mathcal{L} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \quad \text{[see Section 1.3]}$$

Quantised version:

$$\hat{\mathcal{H}} = \hat{\pi}_{(\phi)}\dot{\hat{\phi}} - \hat{\mathcal{L}} = \frac{1}{2}(\partial_0\hat{\phi})^2 + \frac{1}{2}(\vec{\nabla}\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 \quad \text{[see Section 1.4]} \tag{[10]}$$

2.1.5 Canonical commutation relation

Application of Eq.([6]):

$$[\phi(\vec{x},t), \pi(\vec{y},t)]_P = \delta^{(3)}(\vec{x}-\vec{y})\,. \tag{[6]bis}$$

Application of Eq.([7]):

$$[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)] = i\hbar\,\delta^{(3)}(\vec{x}-\vec{y})\,\hat{1}\,. \tag{[7]bis}$$

2.2 Decomposition of the field

2.2.1 Field operator expression

In terms of integrable functions (on $\mathbb{R}$): any fonction can be Fourier decomposed. Hence the field operator $\hat{\phi}(x^\nu)$ solution of Eq.([9]) can be rewritten as,

$$\hat{\phi}(x^\nu) = \frac{1}{(2\pi)^{3/2}} \int d^4p\ \delta(\underline{p^2} - m^2)\hat{A}(p^\mu)e^{-\frac{i}{(\hbar)}p^\mu x_\mu} \tag{[12]}$$

We use the notation $\underline{p^2}$ for the Lorentz square (or Lorentz product): $\underline{p^2} = p^\mu p_\mu$. In this equation, p^μ represents well the physical 4-momentum, as one can be convinced of from the two following considerations:
(i) As in Quantum Mechanics, the Fourier transformation of the wave function (x-representation) leads to the wave function in momentum space.
(ii) One can write the wave function as, $e^{-\frac{i}{(\hbar)}p^\mu x_\mu} = \langle\vec{x}|p^\mu(t)\rangle$, $\langle\vec{x}|p^\mu(t)\rangle$ being eigenfunction of the momentum operator, $\hat{P}^i = -i\hbar\partial_i$, and Hamiltonian, $\hat{H} = i\hbar\partial_0$, with respective eigenvalues $\vec{p}$ and p^0.

Fourier transformation with a sign convention different for $\begin{bmatrix} x^0 \\ p^0 \end{bmatrix}$ and $\begin{bmatrix} x^i \\ p^i \end{bmatrix}$ due to the metric $(+ - - -)$.

The introduction of $\delta(\underline{p}^2 - m^2)$ permits ϕ to satisfy the Klein-Gordon equation:

$$(\Box + m^2)\hat{\phi} = \frac{1}{(2\pi)^{3/2}} \int d^4p \; \delta(\underline{p}^2 - m^2)\hat{A}(p^\mu) \underbrace{(\Box + m^2)e^{-ip^0x^0 + ip^jx^j}}_{(\underbrace{[-ip^0]^2 - \Sigma_i [ip^i]^2}_{-[(p^0)^2 - \Sigma_i (p^i)^2] = -\underline{p}^2} + m^2)e^{-ip^\mu x_\mu}} = 0$$

with $\Box \hat{=} \partial_0\partial_0 - \partial_i\partial_i$.

Hermitian scalar field operator: For a field such that $\hat{\phi} \hat{=} \hat{\phi}^\dagger(x^\nu)$, one has (see Tutorials 2, Exercise 1),

$$\hat{A}^\dagger(p) = \hat{A}(-p)\,. \qquad ([13])$$

Let us now rewrite $\hat{\phi}$ in a useful form, using the step function introduced as follows,

$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 0 & \text{for } x < 0 \end{cases} \Leftrightarrow \Theta(-x') = \begin{cases} 1 & -x' > 0 \\ \frac{1}{2} & -x' = 0 \\ 0 & -x' < 0 \end{cases} \Leftrightarrow \Theta(-x) = \begin{cases} 0 & x > 0 \\ \frac{1}{2} & x = 0 \\ 1 & x < 0 \end{cases}, \forall x$$

so that $\Theta(x) + \Theta(-x) = 1, \forall x$. Introducing this sum in Eq.([12]),

$$\begin{aligned} \hat{\phi}(x^\nu) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^4p \; \delta(\underline{p}^2 - m^2)\Theta(p^0)\hat{A}(p)e^{-ip^\mu x_\mu} \\ &+ \underbrace{\frac{1}{(2\pi)^{\frac{3}{2}}} \int d^4p \; \delta(\underline{p}^2 - m^2)\Theta(-p^0)\hat{A}(p)e^{-ip^\mu x_\mu}}_{\frac{1}{(2\pi)^{3/2}} \int\int\int\int d^4(-p)\delta((-p)^{\underline{2}} - m^2)\Theta(-p^0)\hat{A}(-[-p])e^{i(-p^\mu)x_\mu}} \end{aligned}$$

$$\hat{\phi}(x^\nu) = \frac{1}{(2\pi)^{3/2}} \int d^4p \; \delta(\underline{p}^2 - m^2)\Theta(p^0)\Big[\hat{A}(p)e^{-ip^\mu x_\mu} + \underbrace{\hat{A}(-p)}_{= \hat{A}^\dagger(p) \text{ from Eq.([13])}} e^{ip^\mu x_\mu}\Big]\,. \qquad ([14])$$

For a simpler form $\Rightarrow$ eliminate the "p^0" dependence.
Let us recall the mathematical formula,

$$\delta[f(x)] = \Sigma_n \frac{\delta(x - x_n)}{\left|\frac{df}{dx}\right|_{x = x_n}} \quad \text{for} \quad f(x_n) = 0, \; \left|\frac{df(x)}{dx}\right|_{x = x_n} \neq 0$$

and apply it to

$$f(x) \equiv f(p^0) = (p^0)^2 - \vec{p}^2 - m^2 = p^\mu p_\mu - m^2$$

which is vanishing for $p^0 = \underbrace{\pm\sqrt{\vec{p}^2 + m^2}}_{x_n} \hat{=} \pm E_p$. We take $E_p > 0$ as a convention throughout those lectures.

$$\delta[\underline{p}^2 - m^2] = \frac{\delta(p^0 - E_p)}{|2p_0|_{p_0=E_p}} + \frac{\delta(p^0 + E_p)}{|2p_0|_{p_0=-E_p}} \qquad ([15])$$

Using the $\underline{p}.\underline{x} = p^\mu x_\mu$ notation for the product of Lorentz (with an implicit sum on $\mu = 0, ..., 4$) and inserting Eq.([15]) into Eq.([14]) gives rise to:

$$\hat{\phi} = \frac{1}{(2\pi)^{3/2}} \int d^4p \, \frac{1}{2E_p} \left[\underbrace{\Theta(p_0)}_{=1} \delta(p_0 - E_p) + \underbrace{\Theta(p_0)}_{=0} \delta(p_0 + E_p) \right] \left(\hat{A}(p) e^{-i\underline{x}.\underline{p}} + \hat{A}^\dagger(p) e^{i\underline{x}.\underline{p}} \right)$$

$$\hat{\phi}(x^\nu) = \int d^3p \, \frac{1}{(2\pi)^{3/2} \, 2E_p} \left(\hat{A}(p^\mu) e^{-i\underline{x}.\underline{p}} + \hat{A}^\dagger(p^\mu) e^{i\underline{x}.\underline{p}} \right)$$

$$\boxed{\hat{\phi}(x^\nu) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(\hat{a}(p^\mu) e^{-i\underline{x}.\underline{p}} + \hat{a}^\dagger(p^\mu) e^{i\underline{x}.\underline{p}} \right)} \quad \text{with} \quad \hat{a}(p^\mu) = \frac{\hat{A}(p^\mu)}{\sqrt{2E_p}} \, . \qquad ([16])$$

Remark : in this equality, one has $p_0 (= p^0) = E_p$ ($\Rightarrow \underline{p}^2 = m^2$).

$$\text{Eq.([11])} \rightarrow \boxed{\hat{\pi}_{(\phi)}(x^\nu) = \int d^3p \; i \sqrt{\frac{E_p}{2(2\pi)^3}} \left(- \hat{a}(p^\mu) e^{-i\underline{x}.\underline{p}} + \hat{a}^\dagger(p^\mu) e^{i\underline{x}.\underline{p}} \right)} \qquad ([17])$$

2.2.2 Canonical commutators

Eq.([16]) $\Rightarrow$ $\dim(\hat{a}) \equiv [E]^{-3/2}$.

$$\begin{aligned} [\hat{a}_p, \hat{a}^\dagger_{p'}] &= \delta^{(3)}(\vec{p} - \vec{p}\,') \, \mathbb{1} \\ [\hat{a}_p, \hat{a}_{p'}] &= 0 \\ [\hat{a}^\dagger_p, \hat{a}^\dagger_{p'}] &= 0 \end{aligned} \qquad ([18])$$

$\Big[$ $\Rightarrow$ (done in Tutorials 2, Exercise 2)
$\Leftarrow$ (demonstration: home exercise) $\quad [\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$ (Eq.([7]bis); $\hbar$ set to unity)

2.3 Decomposition of the Hamiltonian

Eq.([10]) leads to:

$$\hat{H}(t) = \int d^3x \left[\frac{1}{2}(\partial_o \hat{\phi})^2 + \frac{1}{2}\vec{\nabla}\hat{\phi}.\vec{\nabla}\hat{\phi} + \frac{1}{2}m^2\hat{\phi}^2 \right]$$

and $\hat{H}$ can then be expressed in terms of $\hat{a}, \hat{a}^\dagger$ (see Tutorials 2, Exercise 3):

$$\hat{H}(t) = \boxed{\hat{H} = \int d^3p \, \frac{E_p}{2} \left(\hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) + \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) \right)} \hat{=} \int d^3p \; \hat{H}_{\vec{p}} \; . \qquad ([19])$$

One has also the following properties (see Tutorials 2, Exercise 4),

$$[\hat{H}, \hat{a}_{p'}] = -E_{p'} \, \hat{a}_{p'} \; , \qquad ([20])$$

$$[\hat{H}, \hat{a}^\dagger_{p'}] = E_{p'} \, \hat{a}^\dagger_{p'} \; . \qquad ([21])$$

3 Fock space

3.1 Recalling the Harmonic oscillator

Let us remind the formalism of the Harmonic oscillator within non-relativistic Quantum mechanics.

3.1.1 1D Harmonic Oscillator (HO)

$$\hat{H} = \frac{1}{2m}(\hat{P}^2 + \underbrace{m^2\omega^2\hat{X}^2}_{\hat{V}(\hat{X})}) \qquad \omega: \text{ pulsation of HO with dimension } T^{-1}$$

Defining the dimensionless operators,

$$\hat{a} = \underbrace{\frac{1}{\sqrt{2m\hbar\omega}}}_{(M(M[V]^2)TT^{-1})^{1/2} = M[V]}(\hat{P} - im\omega\hat{X}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(\hat{P}^{(\dagger)} + im\omega\hat{X}^{(\dagger)})$$

the Hamiltonian can be rewritten as,

$$\hat{H} = \frac{1}{2}\hbar\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \qquad HO.1$$

$$[\hat{X}, \hat{P}] = i\hbar \Rightarrow [\hat{a}, \hat{a}^\dagger] = \mathbb{1} \qquad HO.2$$

$$[\hat{H}, \hat{a}] = \frac{\hbar\omega}{2}[\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger, \hat{a}] = \frac{\hbar\omega}{2}([\hat{a}^\dagger, \hat{a}]\hat{a} + \hat{a}[\hat{a}^\dagger, \hat{a}]) = -\hbar\omega\hat{a}$$

$$\text{...and similarly, } [\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger. \qquad HO.3$$

Notice the dimension $[\hbar\omega] = [E]$.

$$HO.1 \text{ and } HO.2: \; \hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}). \qquad HO.4$$

$$\hat{H}|n>_{\text{To Normalise}} = E_n|n>_{TN} \Rightarrow \begin{cases} \hat{H}\hat{a}|n>_{TN} = (E_n - \hbar\omega)\hat{a}|n>_{TN} \\ \hat{H}\hat{a}^\dagger|n>_{TN} = (E_n + \hbar\omega)\hat{a}^\dagger|n>_{TN} \;\; \text{(creates a quantum)} \\ \quad \text{“still eigenstate with eigenvalue plus a quantum”} \end{cases}$$

Remark : Energy conservation so $\hat{a}, \hat{a}^\dagger$ appear together as e.g. in $HO.4$.

Ground state (of lowest energy) noted $|0>$ (normalised).
It must respect,

$$\hat{a}|0> = 0 \qquad HO.5$$

$HO.4$: $H|0> = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|0> = \frac{\hbar\omega}{2}|0>$

Then $\underbrace{\hat{a}^\dagger|0>}$ has eigenvalue energy $\frac{\hbar\omega}{2}+\hbar\omega=\hbar\omega(\frac{1}{2}+1)$
noted $|1>_{TN}$ as first excitation

and $\underbrace{a^\dagger|1>_{TN}}$ has eigenvalue energy $\hbar\omega(\frac{1}{2}+2)$ TO generalise: $|n>_{TN}$ with energy $\underbrace{\hbar\omega(\frac{1}{2}+n)}_{\text{"}E_n\text{"}}$ $HO.6$
$|2>_{TN}$ since second excitation

Remark : It turns out that there exists no other eigenvalue of $\hat{H}$.

Hence generalising: $|n>_{TN}=(\hat{a}^\dagger)^n|0>$ and after normalisation one gets,

$$|n>=\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0> \qquad HO.7$$

if imposing,

$$<0|0>=1$$

which is always possible to set. It is easy to convince oneself about this by checking it for first states ($<n|n>=1$) but the systematic ($\forall n$) demonstration is longer.

Annihilation/creation operator actions on (initial/final) normalised states $|n>$ can be deduced:

- $\hat{a}^\dagger|n>=\sqrt{(n+1)}\frac{1}{\sqrt{(n+1)!}}\hat{a}^\dagger(\hat{a}^\dagger)^n|0>=\sqrt{n+1}|n+1>$ $\qquad HO.8$
- so $<n|\hat{a}\hat{a}^\dagger|n>=<n|\sqrt{n+1}\,\hat{a}\,|n+1>\Leftrightarrow 1+||\,\hat{a}|n>||^2=(n+1)<n+1|n+1>$ $\Leftrightarrow 1+||N|n-1>||^2=(n+1)\Leftrightarrow N^2=n\Rightarrow N=\sqrt{n}$. and hence,
  $\hat{a}|n>=\sqrt{n}|n-1>$ $\qquad HO.9$

Operator number: $\hat{\mathcal{N}}=\hat{a}^\dagger\hat{a}$, since

$$\hat{\mathcal{N}}|n>=\hat{a}^\dagger\hat{a}|n>=\hat{a}^\dagger\sqrt{n}|n-1>=\sqrt{n}\sqrt{(n-1)+1}|n>=n|n>. \qquad HO.10$$

Orthonormalisation (non-degenerate energy spectrum leading automatically to orthogonal Hamiltonian eigenstates / eigenspaces):

$$<n|n'>=\delta^{nn'}. \qquad HO.11$$

3.1.2 Generalisation to N HO's

Let us extend this formalism for instance to N dimensions or N systems. Then regarding the individual HO Hamiltonian $\hat{H}_i^{(\mathrm{HO})}$ in each Hilbert space $\mathbb{H}_i$, we have, using the compact notation (writing explicitly only the states/operators, of individual Hilbert spaces, being non-trivial),

$$\hat{H} = \Sigma_{i=1}^N \frac{1}{2m_i}(\hat{P}_i^2 + m_i^2\omega_i^2\hat{X}_i^2) = \Sigma_{i=1}^N \hat{H}_i^{(\mathrm{HO})} = \Sigma_{i=1}^N(\ldots \mathbb{1}\otimes\hat{H}_i^{(HO)}\otimes\mathbb{1}\otimes\mathbb{1}\ldots)$$

$$\hat{H} = \Sigma_i\frac{\hbar\omega_i}{2}(\hat{a}_i^\dagger\hat{a}_i + \hat{a}_i\hat{a}_i^\dagger) \qquad HO.1b(is)$$

$$\hat{H} = \Sigma_i\frac{\hbar\omega_i}{2}(2\,\hat{a}_i^\dagger\hat{a}_i + \mathbb{1}). \qquad HO.4b$$

Regarding the states,

$$\hat{a}_i|0>_{(i)} = 0 \qquad HO.5b$$

$$\hat{H}_i|n_i>= E_{n_i}|n_i>\,,\ E_{n_i} = \hbar\omega_i(\frac{1}{2}+n_i) \qquad HO.6b$$

$$|n_i>= \frac{1}{\sqrt{n_i!}}(\hat{a}_i^\dagger)^{n_i}|0>_{(i)} \qquad HO.7b$$

$$\hat{a}_i^\dagger|n_i>= \sqrt{n_i+1}|n_i+1> \qquad HO.8b$$

$$\hat{a}_i|n_i>= \sqrt{n_i}|n_i-1>\,. \qquad HO.9b$$

Regarding the annihilation/creation operators,

$$[\hat{a}_i,\hat{a}_j^\dagger] = \delta^{ij}\mathbb{1}\,,\ [\hat{a}_i,\hat{a}_j] = 0\,,\ [\hat{a}_i^\dagger,\hat{a}_j^\dagger] = 0 \qquad HO.2b$$

$$[\hat{H},\hat{a}_i] = -\hbar\omega_i\hat{a}_i\,,\ [\hat{H},\hat{a}_i^\dagger] = \hbar\omega_i\hat{a}_i^\dagger. \qquad HO.3b$$

Demonstration:

$$[\hat{H},\hat{a}_j] = [\Sigma_i\frac{\hbar\omega_i}{2}(\hat{a}_i^\dagger\hat{a}_i + \hat{a}_i\hat{a}_i^\dagger),\hat{a}_j] = \frac{\hbar\omega_j}{2}[(\hat{a}_j^\dagger\hat{a}_j + \hat{a}_j\hat{a}_j^\dagger),\hat{a}_j] = -\hbar\omega_j\hat{a}_j$$

since $\hat{a}_j$ commutes with all $\hat{a}_i$ for $i \neq j$ as they belong to different $\mathcal{H}$'s.

Same individual energy spectrum (HO.6). Globally, $\hat{H}$ has an additive form ("separate variables" case as each term of the sum is non-trivial in a given subspace, without subspace mixing terms) so its energy eigenvalues and associated eigenstates are,

$$E = \Sigma_{i=1}^N E_{n_i} = \Sigma_{i=1}^N\hbar\omega_i(\frac{1}{2}+n_i) = \Sigma\frac{\hbar\omega_i}{2} + \Sigma\hbar\omega_i n_i$$

$$|E>=|n_1>\otimes|n_2>\otimes\ldots\otimes|n_N> \quad \text{(set of } n_i \text{ values).} \qquad (16)$$

Example n'1: $n_1 = 2$ (all the other n_i vanish) and common pulsation,

$$\hat{H}\ |2>_{(1)}\otimes|0>_{(2)}\otimes\ldots\otimes|0>_{(N)}$$

$$= \left[\hbar\omega(\hat{a}_1^\dagger\hat{a}_1+\frac{1}{2})\otimes\mathbb{1}\otimes\ldots\otimes\mathbb{1} + \mathbb{1}\otimes\hat{H}_2^{(\mathrm{HO})}\otimes\ldots\otimes\mathbb{1} + \ldots + \mathbb{1}\otimes\ldots\otimes\mathbb{1}\otimes\hat{H}_N^{(\mathrm{HO})}\right]$$

$$|2>_{(1)}\ |0>_{(2)}\ \ldots$$

$$= \left(\hbar\omega(2+\frac{1}{2}) + \frac{\hbar\omega}{2} + \ldots\right)\ |2>_{(1)}\ |0>_{(2)}\ \ldots$$

$$= \hbar\omega(2+\frac{N}{2})\ |2>_{(1)}\ |0>_{(2)}\ \ldots$$

$$= E\ |2>_{(1)}\ |0>_{(2)}\ \ldots$$

Example n'2: $n_1 = 2$, $n_2 = 3$ (all the other n_i vanish) and common pulsation,

$$\hat{H}\ |2>_{(1)}\ |3>_{(2)}\ |0>_{(3)}\ ...$$
$$= \hbar\omega(2+3+\frac{N}{2})\ |2>_{(1)}\ |3>_{(2)}\ |0>_{(3)}\ ... = E\ |2>_{(1)}\ |3>_{(2)}\ |0>_{(3)}\ ...$$

Finally,

$$\hat{\mathcal{N}} = \Sigma_i \hat{\mathcal{N}}_i = \Sigma_i \hat{a}_i^\dagger \hat{a}_i\ ,\ \text{with,}\ \ \hat{\mathcal{N}}_i |n_i>= n_i |n_i> \qquad HO.10b$$
$$< n_i|n'_j >= \delta^{n_i n'_j} \delta^{ij}\ ,\ \text{with,}\ \ _{(i)} < 0|0 >_{(j)} = 1. \qquad HO.11b$$

Indeed, let us consider an example with $i = j = 2$,

$$< n_2|n'_2 >= \Big(_{(1)} < 0|\ \otimes\ _{(2)} < n|\ \otimes\ \dots\Big)\Big(|0>_{(1)}\ \otimes |n'>_{(2)}\ \otimes\ \dots\Big)$$
$$=< 0|0 > \times < n_2|n'_2 > \times \cdots = 1 \times \delta^{n_2 n'_2} \times \cdots = \delta^{n_2 n'_2}\ ,$$

an another one with $i \neq j$ and $n_i \neq 0$, $n'_j \neq 0$,

$$< n_1|n'_2 >= \Big(_{(1)} < n|\ \otimes\ _{(2)} < 0|\ \otimes\ \dots\Big)\Big(|0>_{(1)}\ \otimes |n'>_{(2)}\ \otimes\ \dots\Big)$$
$$=< n_1|0 > \times < 0|n'_2 > \times \cdots = 0 \times 0 \times \cdots = 0\,.$$

Notice that multi-space state scalar products, like for instance

$$(< n_i|\ < n'_j|)\ (|n''_k >\ |n'''_\ell >),$$

can be directly deduced from those basic calculations.

3.2 Multi-particle states

3.2.1 Analogy between the HO's and quantised scalar field

Following a mathematical analogy, a real free scalar field system (associating one Hilbert space $\mathbb{H}_{\vec{p}}$ to each $\vec{p}$) can be seen formally as a continuous version of an infinite multi-HO (taking the limiting case $N \to \infty$):

$$\begin{cases} \hat{H}_i \text{ on } \mathbb{H}_i,\ \hat{a}_i,\ \hbar\omega_i,\ n_i \mapsto \hat{H}_{\vec{p}} \text{ on } \mathbb{H}_{\vec{p}},\ \hat{a}_{\vec{p}},\ E_p,\ n_{\vec{p}} \\ HO.1b \mapsto \text{Eq.([19])} \\ HO.2b \mapsto \text{Eq.([18])}\ \ \text{(implicit compact notation then for multi-Hilbert spaces)} \\ HO.3b \mapsto \text{Eq.([20])} - \text{([21])} \\ HO.10b \mapsto \hat{\mathcal{N}} = \int \hat{\mathcal{N}}_{\vec{p}}\, d^3p = \int \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}}\, d^3p \\ HO.1b\ \&\ HO.2b \Rightarrow HO.4b \mapsto \text{Eq.([18])}\ \&\ \text{Eq.([19])} \Rightarrow \text{Eq.([22]bis)}\ \ \text{(see below)} \end{cases}$$

Therefore, as in the multi-HO formalism where the relations *HO*.1b-4b, 10b induce the relations *HO*.5b-9b, 11b, (16) [1st quantisation], the above relations on the field *operators* induce similar relations on the individual energy *eigenstates* [2nd quantisation: relativistic with wave function variables] provided here and in the two following subsections [$\sqrt{(2\pi)^3/V}$ factor justified by a constant state dimension for any level given that $\dim(\hat{a}^{(\dagger)}) \equiv [E]^{-3/2}$ as stated in Section 2.2.2 [1]]:

$HO.5b \mapsto$

$$\left(\sqrt{\frac{(2\pi)^3}{V}}\,\hat{a}_{\vec{p}}\right)\,|0>_{(\vec{p})}=0\,,\quad \text{or,}\quad <0|_{(\vec{p})}\left(\sqrt{\frac{(2\pi)^3}{V}}\hat{a}^{\dagger}_{\vec{p}}\right)=0 \tag{17}$$

$HO.7b \mapsto$

$$|n_{\vec{p}}>=\frac{1}{\sqrt{n_{\vec{p}}!}}\left(\sqrt{\frac{(2\pi)^3}{V}}\,\hat{a}^{\dagger}_{\vec{p}}\right)^{n_{\vec{p}}}\,|0>_{(\vec{p})} \tag{18}$$

$HO.8b \mapsto$

$$\sqrt{\frac{(2\pi)^3}{V}}\,\hat{a}^{\dagger}_{\vec{p}}\,|n_{\vec{p}}>=\sqrt{n_{\vec{p}}+1}\,|n_{\vec{p}}+1> \tag{19}$$

$HO.9b \mapsto$

$$\sqrt{\frac{(2\pi)^3}{V}}\,\hat{a}_{\vec{p}}\,|n_{\vec{p}}>=\sqrt{n_{\vec{p}}}\,|n_{\vec{p}}-1>\,. \tag{20}$$

One has to define any state within a finite volume V, and then take the limit $V \to \infty$ at the end of calculation: it constitutes a physical regularisation process, mathematically allowed. As a matter of fact, regarding the orthonormalisation condition, one has for instance (using compact space notation),

$$\begin{aligned}
{}_{(\vec{p})}<1|1>_{(\vec{p}')} &= \frac{(2\pi)^3}{V}{}_{(\vec{p})}<0|\hat{a}_p\,\hat{a}^{\dagger}_{p'}|0>_{(\vec{p}')}=\frac{(2\pi)^3}{V}<0|\delta^{(3)}(\vec{p}-\vec{p}')+\hat{a}^{\dagger}_{p'}\hat{a}_p|0> \\
&= \frac{(2\pi)^3}{V}\delta^{(3)}(\vec{p}-\vec{p}')\underbrace{<0|0>}_{=1}\underbrace{=}_{\text{if } \vec{p}=\vec{p}'}\frac{(2\pi)^3}{V}\times\lim_{V\to\infty}\left[\frac{1}{(2\pi)^3}\int_V d^3x\;e^{-i\,\vec{0}_{\vec{p}}.\vec{x}}\,1_x\right]=1\,, \\
{}_{(\vec{p})}<1|2>_{(\vec{p}')} &= \left(\frac{(2\pi)^3}{V}\right)^{3/2}<0|\hat{a}_p\frac{1}{\sqrt{2!}}\hat{a}^{\dagger}_{p'}\,\hat{a}^{\dagger}_{p'}|0> \\
&= \left(\frac{(2\pi)^3}{V}\right)^{3/2}\frac{1}{\sqrt{2}}<0|(\delta^{(3)}(\vec{p}-\vec{p}')+\hat{a}^{\dagger}_{p'}\hat{a}_p)\hat{a}^{\dagger}_{p'}|0>=0\,,
\end{aligned}$$

so that,

$$<n_{\vec{p}}|n'_{\vec{p}'}>=\left(\frac{(2\pi)^3}{V}\right)\delta^{n_{(\vec{p})}n'_{(\vec{p}')}}\delta^{(3)}(\vec{p}-\vec{p}')\,, \tag{21}$$

which extends *HO*.11b. Notice then the induced dimensional analysis for the quantum states: $[|n_{\vec{p}}>]=1$.

[1] In order to build dimensionless operators like in the original HO case of Section 3.1.1. Indeed, a dimension factor is needed due to the change from a discrete to a continuous spectrum (like $\sum_i \mapsto \int d^3p$).

3.2.2 Normal ordering

This part addresses a subtlety due to the continuous set of HO's. Let us rewrite $\hat{H}$ again using the $a^{(\dagger)}$ commutation relation:
$HO.4$b $\mapsto$

$$\text{Eq.([19])} \Rightarrow \hat{H} = \int d^3p\ \frac{E_p}{2}(\hat{a}_p\hat{a}_p^\dagger + \hat{a}_p^\dagger\hat{a}_p) \underbrace{=}_{\text{Eq.([18]) with compact notation for } ...\otimes\mathbb{1}\otimes\hat{a}_p\otimes\mathbb{1}\otimes...} \int d^3p\ \frac{E_p}{2}(2\hat{a}_p^\dagger\hat{a}_p + \delta^{(3)}(\vec{0}_p)\,\mathbb{1}) \quad \text{([22]bis)}$$

$\delta^{(3)}(\vec{0}_p)$ is undefined mathematically but also belongs anyway to the term that physically accounts for the problematic infinite contribution of finite ground state energies, as in $HO.4$b: $\Sigma_{i=1}^{\infty}\frac{\hbar\omega_i}{2}\mathbb{1} \in \hat{H}$ (whereas $\hat{a}_i^\dagger\hat{a}_i$ gives rise to zero through its action on $|0>$). See also Eq.(16).

The way out: only energy differences are physical (as classically: $\vec{F} = -\vec{\nabla}V$) so one can introduce a prescription redefining the energy absolute scale. This is the so-called "Normal ordering rearrangement".

Apply to $\hat{H} \equiv$ {
1.- Write all annihilation operators to the right of creation operators in products, as if they were commuting with each other.
2.- Treat $\hat{a}$ and $\hat{a}^\dagger$ with usual commutation rules of Eq.([18]), again.
}

Here it gives (for the step 1),

$$:\hat{H}: = \int d^3p\ \frac{Ep}{2}(\hat{a}_p^\dagger\hat{a}_p + \underbrace{:\hat{a}_p\hat{a}_p^\dagger:}_{\hat{a}_p^\dagger\hat{a}_p}) \ = \int d^3p\ E_p\,\hat{a}_p^\dagger\hat{a}_p \ = \int d^3p\ E_p\hat{\mathcal{N}}_{\vec{p}} \hat{=} \int d^3p\ :\hat{H}_{\vec{p}}: \quad \text{([22])}$$

which has indeed redefined the ground energy in a minimal way, from Eq.([19]) to ([22]bis), only eliminating the $\delta^{(3)}(\vec{0})$ term (same $a^{(\dagger)}$ formalism):

$$:\hat{H}:|0>= \int d^3p\,E_p\,\hat{a}_p^\dagger\hat{a}_p|0>= 0 = 0|0>= E_0|0>.$$

<u>Remark :</u>

$$<\Psi|:\hat{H}:|\Psi>= \int d^3p\ <\Psi|\hat{a}_p^\dagger\hat{a}_p|\Psi> E_p = \int d^3p\ E_p\ ||\,\hat{a}_p|\Psi>||^2 \geq 0.$$

(recall that E_p is positive by definition) This confirms that $E_0 = 0$ is well the ground/minimal energy: $<\Psi|:\hat{H}:|\Psi>\geq E_{\min} = E_0 = 0$.

$HO.6b \mapsto$

$$:\hat{H}_{\vec{p}}:|n_{\vec{p}}>= \hat{\mathcal{N}}_{\vec{p}}\,E_p|n_{\vec{p}}>= \hat{a}_{\vec{p}}^\dagger\hat{a}_{\vec{p}}\,E_p\,|n_{\vec{p}}> = E_{n_{\vec{p}}}\,\delta^{(3)}(\vec{0}_{\vec{p}})|n_{\vec{p}}>,\ E_{n_{\vec{p}}} = n_{\vec{p}}\,E_p\ . \quad (22)$$

Demonstration (see another one in the Tutorials 2, Exercise 5): based on Eq.(19)-(20) and $V/(2\pi)^3 \equiv \delta^{(3)}(\vec{0}_{\vec{p}})$ [as seen above Eq.(21) through the regularisation process], we can write,

$$\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\,E_p\,|n_{\vec{p}}>= E_p\,\hat{a}^\dagger_{\vec{p}}\sqrt{\frac{V}{(2\pi)^3}}\,\sqrt{n_{\vec{p}}}\,|n_{\vec{p}}-1>= E_p\,\frac{V}{(2\pi)^3}\,n_{\vec{p}}\,|n_{\vec{p}}> \ .$$

More generally (see the Tutorials 2, Exercise 5, for the case $n_{\vec{p}}=1$),

$$:\hat{H}_{\vec{p}}:|n_{\vec{p'}}>= \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\,E_p\,|n_{\vec{p'}}> = E_{n_{\vec{p}}}\,\delta^{(3)}(\vec{p}-\vec{p}\,')|n_{\vec{p}}>,\ E_{n_{\vec{p}}} = n_{\vec{p}}\,E_p\ , \qquad (23)$$

or equivalently,

$$\hat{\mathcal{N}}_{\vec{p}}\,|n_{\vec{p'}}> = n_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{p}\,')|n_{\vec{p}}> \ .$$

Remark : Eq.([20]) - Eq.([21]) remain true for : $\hat{H}$: (global factor 2 in Eq.([22]) compensates one term less: $\hat{a}_p\hat{a}^\dagger_p$, compared to Eq.([19])).

3.2.3 Physical/Quantum interpretation

For example,

$$:\hat{H}:|E> = \left(\int d^3p\ E_p(...\otimes \mathbb{1}\otimes \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\otimes \mathbb{1}_{\vec{p}+\delta\vec{p}}\otimes ...)\right)(...|0>\otimes|2>_{(\vec{p}_1)}\otimes|0>_{(\vec{p}_1+\delta\vec{p}_1)}\otimes|0>\otimes...)$$

$$= \int d^3p\ E_p\ \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\ |2>_{(\vec{p}_1)} = \int d^3p\ E_p\ \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\ \left(\frac{1}{\sqrt{2!}}\left(\sqrt{\frac{(2\pi)^3}{V}}\,\hat{a}^\dagger_{\vec{p}_1}\right)^2 |0>_{(\vec{p}_1)}\right)$$

$$= \int d^3p\ E_p\ \frac{1}{\sqrt{2!}}\,\hat{a}^\dagger_{\vec{p}}\left(\hat{a}^\dagger_{\vec{p}_1}\,\hat{a}_{\vec{p}} + \delta^{(3)}(\vec{p}-\vec{p}_1)\right)\hat{a}^\dagger_{\vec{p}_1}\left(\sqrt{\frac{(2\pi)^3}{V}}\right)^2 |0>_{(\vec{p}_1)}$$

$$= \int d^3p\ E_p\ \frac{1}{\sqrt{2!}}\,\hat{a}^\dagger_{\vec{p}}\left(\hat{a}^\dagger_{\vec{p}_1}\,[\hat{a}^\dagger_{\vec{p}_1}\hat{a}_{\vec{p}} + \delta^{(3)}(\vec{p}-\vec{p}_1)] + \delta^{(3)}(\vec{p}-\vec{p}_1)\hat{a}^\dagger_{\vec{p}_1}\right)\left(\sqrt{\frac{(2\pi)^3}{V}}\right)^2 |0>_{(\vec{p}_1)}$$

$$= \int d^3p\ E_p\ \frac{1}{\sqrt{2!}}\,2\,\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{p}_1}\,\delta^{(3)}(\vec{p}-\vec{p}_1)\left(\sqrt{\frac{(2\pi)^3}{V}}\right)^2 |0>_{(\vec{p}_1)}$$

$$= \int d^3p\ E_p\ \frac{1}{\sqrt{2!}}\,2\,(\hat{a}^\dagger_{\vec{p}})^2\,\delta^{(3)}(\vec{p}-\vec{p}_1)\left(\sqrt{\frac{(2\pi)^3}{V}}\right)^2 |0>_{(\vec{p}_1)}$$

$$= \int d^3p\,\delta^{(3)}(\vec{p}-\vec{p}_1)\ 2\,E_p\ \left(\frac{1}{\sqrt{2!}}\,(\hat{a}^\dagger_{\vec{p}})^2\left(\sqrt{\frac{(2\pi)^3}{V}}\right)^2 |0>\right)$$

$$= \int d^3p\,\delta^{(3)}(\vec{p}-\vec{p}_1)\ 2\,E_p\ |2>_{(\vec{p})} = 2\,E_{p_1}\ |2>_{(\vec{p}_1)} = \underbrace{2\,E_{p_1}}_{\text{whole energy}}\ |E>$$

with $n_{\vec{p}_1} = 2$. Another example:

$$
\begin{aligned}
:\hat{H}:|E> &= \int d^3p\ E_p\ \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\ \left(...|0> \otimes |1>_{(\vec{p}_1)} \otimes |1>_{(\vec{p}_1{}')} \otimes |0> \otimes ...\right) \\
&= \int d^3p \left(\sqrt{\frac{(2\pi)^3}{V}}\right)^2 E_p\ \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} \\
&\qquad \left(...|0> \otimes \left[\frac{1}{\sqrt{1!}}\ \hat{a}^\dagger_{\vec{p}_1}|0>_{(\vec{p}_1)}\right] \otimes \left[\frac{1}{\sqrt{1!}}\ \hat{a}^\dagger_{\vec{p}_1{}'}|0>_{(\vec{p}_1{}')}\right] \otimes |0> \otimes ...\right) \\
&= \int d^3p \left(\sqrt{\frac{(2\pi)^3}{V}}\right)^2 E_p\ \left(...\mathbb{1} \otimes \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}(\otimes)\hat{a}^\dagger_{\vec{p}_1} \otimes \hat{a}^\dagger_{\vec{p}_1{}'} \otimes \mathbb{1} \otimes ...\right) \\
&\qquad \left(...|0> \otimes |0>_{(\vec{p}_1)} \otimes |0>_{(\vec{p}_1{}')} \otimes |0> \otimes ...\right) \\
&= \int d^3p \left(\sqrt{\frac{(2\pi)^3}{V}}\right)^2 E_p\ \hat{a}^\dagger_{\vec{p}}\ \left(\hat{a}^\dagger_{\vec{p}_1}\ \hat{a}_{\vec{p}} + \delta^{(3)}(\vec{p}-\vec{p}_1)\right)\ \hat{a}^\dagger_{\vec{p}_1{}'}|0> \\
&= \int d^3p \left(\sqrt{\frac{(2\pi)^3}{V}}\right)^2 E_p\ \left(\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{p}_1}\ [\hat{a}^\dagger_{\vec{p}_1{}'}\hat{a}_{\vec{p}} + \delta^{(3)}(\vec{p}-\vec{p}_1{}')] + \hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{p}_1{}'}\delta^{(3)}(\vec{p}-\vec{p}_1)\right)\ |0> \\
&= \int d^3p\ E_p\ \delta^{(3)}(\vec{p}-\vec{p}_1{}')\ |1>_{(\vec{p})}\ |1>_{(\vec{p}_1)} + \int d^3p\ E_p\ \delta^{(3)}(\vec{p}-\vec{p}_1)\ |1>_{(\vec{p})}\ |1>_{(\vec{p}_1{}')} \\
&= E_{p_1{}'}\ |1>_{(\vec{p}_1{}')}\ |1>_{(\vec{p}_1)} + E_{p_1}\ |1>_{(\vec{p}_1)}\ |1>_{(\vec{p}_1{}')} = \underbrace{(E_{p_1} + E_{p_1{}'})}_{\text{whole energy}}\ |E>
\end{aligned}
$$

with $n_{\vec{p}_1} = n_{\vec{p}_1{}'} = 1$. Generalisation (for a finite number of non-vanishing $n_{\vec{k}}$):

$$
:\hat{H}:\ (|n_{\vec{p}}> \otimes |n_{\vec{p}\,'}> \otimes ...) = \left(\sum_{\vec{k}=\vec{p},\vec{p}\,',...} n_{\vec{k}}\, E_k\right)\ (|n_{\vec{p}}> \otimes |n_{\vec{p}\,'}> \otimes ...) \tag{24}
$$

which extends Eq.(16) (and Eq.(22)). Full generalisation (continuous case):

$$
:\hat{H}:\ (\Pi^{\text{cont.}}_{\vec{k}}|_{\otimes}\ |n_{\vec{k}}>) = \left(\frac{V}{(2\pi)^3}\int d^3k\ n_{\vec{k}}\, E_k\right)\ (\Pi^{\text{cont.}}_{\vec{k}}|_{\otimes}\ |n_{\vec{k}}>)\,. \tag{25}
$$

Those results allow to interpret physically $n_{\vec{k}}$ as being the number of particles with energy E_k, and, to give a symbolic simplified picture of the whole **Fock space** $\mathbb{F}$:

$$
\left[\mathbb{H}^{\text{part.1}} \otimes \left(\mathbb{H}^{\text{part.2}}_{\hat{X},\hat{P}_X} \otimes \mathbb{H}^{\text{part.2}}_{\hat{Y},\hat{P}_Y} \otimes \mathbb{H}^{\text{part.2}}_{\hat{S}_x} ...\right) ...\right]
$$
$$
\otimes \left[\left(\mathbb{H}^{\text{part.1}}_{\vec{p}} \otimes \mathbb{H}^{\text{part.1}}_{\vec{p}\,'} \otimes ...\right) \otimes \left(\mathbb{H}^{\text{part.2}}_{\vec{q}} \otimes \mathbb{H}^{\text{part.2}}_{\vec{q}\,'} \otimes ...\right) ...\right]
$$

4 Complex scalar field

4.1 Lagrangian

The Lagrangian of the complex scalar field operator $[\hat{\phi}^\dagger \neq \hat{\phi}]$ reads as,

$$\boxed{\hat{\mathcal{L}} = \partial_\mu \hat{\phi}^\dagger \partial^\mu \hat{\phi} - m^2 \hat{\phi}^\dagger \hat{\phi}} \qquad ([23])$$

$$\underbrace{\Rightarrow}_{EOM(\hat{\phi},\hat{\phi}^\dagger)} \quad \text{K.-G. equation (for } \hat{\phi} \text{ and } \hat{\phi}^\dagger)$$

Remark : $\hat{\mathcal{L}}^\dagger = \hat{\mathcal{L}}$.

Defining two real fields (and then two Hermitian field operators) as,

$$\hat{\phi}(x^\nu) \hat{=} \frac{1}{\sqrt{2}} \left(\hat{\phi}_1(x^\nu) + i\hat{\phi}_2(x^\nu) \right)$$

Eq.([23]) can be re-expressed as,

$$\hat{\mathcal{L}} = \frac{1}{2}(\partial_\mu \hat{\phi}_1 + i\partial_\mu \hat{\phi}_2)^\dagger (\partial^\mu \hat{\phi}_1 + i\partial^\mu \hat{\phi}_2) - m^2 \frac{1}{2}(\hat{\phi}_1 + i\hat{\phi}_2)^\dagger (\hat{\phi}_1 + i\hat{\phi}_2)$$

$$= \frac{1}{2}[\partial_\mu \hat{\phi}_1^{(\dagger)} \partial^\mu \hat{\phi}_1 + i\partial_\mu \hat{\phi}_1^{(\dagger)} \partial^\mu \hat{\phi}_2 - i\partial_\mu \hat{\phi}_2^{(\dagger)} \partial^\mu \hat{\phi}_1 + \partial_\mu \hat{\phi}_2^{(\dagger)} \partial^\mu \hat{\phi}_2] - \frac{m^2}{2}[\hat{\phi}_1^{(\dagger)} \hat{\phi}_1 + i\hat{\phi}_1^{(\dagger)} \hat{\phi}_2 - i\hat{\phi}_2^{(\dagger)} \hat{\phi}_1 + \hat{\phi}_2^{(\dagger)} \hat{\phi}_2] \, .$$

Since $[\hat{\phi}_1, \hat{\phi}_2] = 0$ (as seen later on), we can re-write it as,

$$\boxed{\hat{\mathcal{L}} = \Sigma_{i=1}^2 \left[\frac{1}{2} \, \partial_\mu \hat{\phi}_i \, \partial^\mu \hat{\phi}_i - \frac{m^2}{2} \, \hat{\phi}_i^2 \right]} \qquad ([24])$$

as in Eq.([8]) so that one gets the K.-G. Eq.([9]) for both the real fields $\hat{\phi}_1$ and $\hat{\phi}_2$. So the choices of physical degrees of freedom $\hat{\phi}, \hat{\phi}^\dagger$ or $\hat{\phi}_1, \hat{\phi}_2$ are equivalent at this level of free framework.

4.2 Field decomposition

Then repeating the $\hat{\phi}_{1,2}$ decomposition based on Eq.([12]), one finds identical commutation relations as in Eq.([18]):

$$[\hat{a}_{ip}, \hat{a}_{ip'}^\dagger] = \delta^{(3)}(\vec{p'} - \vec{p}), \quad [\hat{a}_{ip}, \hat{a}_{ip'}] = [\hat{a}_{ip}^\dagger, \hat{a}_{ip'}^\dagger] = 0, \quad \text{with i=1,2, and,} \quad [\hat{a}_{1p}^{(\dagger)}, \hat{a}_{2p^{(\prime)}}^{(\dagger)}] = 0, \qquad ([18]\text{bis})$$

Indeed, based on Eq.([10]) and Eq.(12), $\hat{\mathcal{L}} = \hat{\mathcal{L}}_1(\hat{\phi}_1) + \hat{\mathcal{L}}_2(\hat{\phi}_2) \Rightarrow \hat{\mathcal{H}} = \hat{\mathcal{H}}_1(\hat{\phi}_1) + \hat{\mathcal{H}}_2(\hat{\phi}_2)$. Then, given the two distinct fields/particles, one is within the quantum separate variable configuration so that one individual Hilbert space $\mathbb{H}_i$ is introduced for each of the two $\hat{\phi}_{1,2}$ fields. There are thus commuting with each other.

Let us now define the operators,

$$\begin{cases} \hat{a}_p \hat{=} \frac{1}{\sqrt{2}}(\hat{a}_{1p} + i\hat{a}_{2p}) \\ \hat{\tilde{a}}_p \hat{=} \frac{1}{\sqrt{2}}(\hat{a}_{1p} - i\hat{a}_{2p}) \end{cases} \Leftrightarrow \begin{cases} \hat{a}_p^\dagger \hat{=} \frac{1}{\sqrt{2}}(\hat{a}_{1p}^\dagger - i\hat{a}_{2p}^\dagger) \\ \hat{\tilde{a}}_p^\dagger \hat{=} \frac{1}{\sqrt{2}}(\hat{a}_{1p}^\dagger + i\hat{a}_{2p}^\dagger) \end{cases}$$

in order to write $\hat{\phi} = (\hat{\phi}_1 + i\hat{\phi}_2)/\sqrt{2}$ (factor for the fields and $H, \hat{H}$ normalisations) in a compact form:

$$\hat{\phi} = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} (\hat{a}_{1p} e^{-ix.p} + \hat{a}_{1p}^\dagger e^{ix.p} + i[\hat{a}_{2p} e^{-ix.p} + \hat{a}_{2p}^\dagger e^{ix.p}]) \frac{1}{\sqrt{2}}$$

$$\boxed{\hat{\phi}(x^\mu) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} (\hat{a}_p e^{-ix.p} + \hat{\tilde{a}}_p^\dagger e^{ix.p})} \Rightarrow \hat{\phi}^\dagger = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} (\hat{\tilde{a}}_p e^{-ix.p} + \hat{a}_p^\dagger e^{ix.p}) \tag{26}$$

which is comparable to Eq.([16]). We show that, as/from Eq.([18]bis),

$$\left.\begin{aligned} &[\hat{a}_p, \hat{a}_{p'}^\dagger] = \frac{1}{2}[\hat{a}_{1p} + i\hat{a}_{2p}, \hat{a}_{1p'}^\dagger - i\hat{a}_{2p'}^\dagger] = \delta^{(3)}(\vec{p} - \vec{p}')\,\mathbb{1} \\ &[\hat{a}_p, \hat{a}_{p'}] = [\hat{a}_p^\dagger, \hat{a}_{p'}^\dagger] = 0 \\ &[\hat{\tilde{a}}_p, \hat{\tilde{a}}_{p'}^\dagger] = \frac{1}{2}[\hat{a}_{1p} - i\hat{a}_{2p}, \hat{a}_{1p'}^\dagger + i\hat{a}_{2p'}^\dagger] = \delta^{(3)}(\vec{p} - \vec{p}')\,\mathbb{1} \\ &[\hat{\tilde{a}}_p, \hat{\tilde{a}}_{p'}] = [\hat{\tilde{a}}_p^\dagger, \hat{\tilde{a}}_{p'}^\dagger] = 0 \end{aligned}\right\} \quad \text{and} \quad [\hat{a}_p^{(\dagger)}, \hat{\tilde{a}}_{p'}^{(\dagger)}] = 0 \qquad ([25])$$

since,

$$[\hat{a}_p, \hat{\tilde{a}}_{p'}] = \frac{1}{2}[\hat{a}_{1p} + i\hat{a}_{2p}, \hat{a}_{1p'} - i\hat{a}_{2p'}] = 0 = [\hat{a}_p^\dagger, \hat{\tilde{a}}_{p'}^\dagger]\ ,$$

$$[\hat{a}_p, \hat{\tilde{a}}_{p'}^\dagger] = \frac{1}{2}[\hat{a}_{1p} + i\hat{a}_{2p}, \hat{a}_{1p'}^\dagger + i\hat{a}_{2p'}^\dagger] = 0 = [\hat{a}_p^\dagger, \hat{\tilde{a}}_{p'}]\ .$$

4.3 Hamiltonian

Eq.([19]) is extending to,

$$\hat{H} = \frac{1}{2} \int d^3p\ E_p \left[(\hat{a}_{1p}\hat{a}_{1p}^\dagger + \hat{a}_{1p}^\dagger \hat{a}_{1p}) + (\hat{a}_{2p}\hat{a}_{2p}^\dagger + \hat{a}_{2p}^\dagger \hat{a}_{2p}) \right].$$

We can easily demonstrate that (see Tutorials 3, Exercise 1(b)):

$$\hat{H} = \frac{1}{2} \int d^3p\ E_p\ (\hat{a}_p\hat{a}_p^\dagger + \hat{a}_p^\dagger\hat{a}_p + \hat{\tilde{a}}_p\hat{\tilde{a}}_p^\dagger + \hat{\tilde{a}}_p^\dagger\hat{\tilde{a}}_p)\,, \qquad ([26]\text{bis})$$

$$: \hat{H} : = \int d^3p \; E_p \; (\hat{a}_p^\dagger \hat{a}_p + \hat{\tilde{a}}_p^\dagger \hat{\tilde{a}}_p) = \int d^3p \; E_p \, (\hat{\mathcal{N}}_{\vec{p}} + \hat{\tilde{\mathcal{N}}}_{\vec{p}}) \,, \tag{[26]}$$

which is similar to Eq.([22]).

Eq.([26]bis) [comparable to Eq.([19])] & Eq.([25]) [comparable to Eq.([18])]
$\Rightarrow$ $[: \hat{H} :, \hat{a}_{p'}] = -E_{p'} \hat{a}_{p'}$ [comparable to Eq.([20])]
and $[: \hat{H} :, \hat{a}_{p'}^\dagger] = E_{p'} \hat{a}_{p'}^\dagger$ [comparable to Eq.([21])],
since $[\hat{\tilde{a}}_p^{(\dagger)}, \hat{a}_{p'}] = 0$ from Eq.([25]). So exhaustively, we have (from similar considerations on the $\hat{\tilde{a}}_p$ operators),

$$[: \hat{H} :, \hat{a}_{p'}] = -E_{p'} \hat{a}_{p'}, \; [: \hat{H} :, \hat{a}_{p'}^\dagger] = E_{p'} \hat{a}_{p'}^\dagger, \; [: \hat{H} :, \hat{\tilde{a}}_{p'}] = -E_{p'} \hat{\tilde{a}}_{p'}, \; [: \hat{H} :, \hat{\tilde{a}}_{p'}^\dagger] = E_{p'} \hat{\tilde{a}}_{p'}^\dagger \;. \tag{[27]}$$

4.4 Quantum states

• Summation form in Eq.([26]) and one term for each kind of particle
$\Rightarrow$ separate variable situation with 2 Hilbert spaces $\mathbb{H}$ (for $\hat{a}_p$) and $\tilde{\mathbb{H}}$ (for $\hat{\tilde{a}}_p$).

• Similar Lagrangians [Eq.([23])], fields [Eq.(26)], Hamiltonian forms [Eq.([26])-([26]bis)] and commutation relations [Eq.([25]), ([27])] for the real scalar field $[\hat{\phi}(\hat{a}_p)]$ and complex scalar field $[\hat{\phi}(\hat{a}_p, \hat{\tilde{a}}_p)]$, that is, generally speaking, for the *operator* structure
$\Rightarrow$ same constructions as in the real field Equations (17)-(21) and (24), also for the new *quantum states* associated to the complex scalar field:

$$\sqrt{\frac{(2\pi)^3}{V}} \; \hat{\tilde{a}}_{\vec{p}} \, |\tilde{0} >_{(\vec{p})} = 0 \tag{27}$$

$$|\tilde{n}_{\vec{p}} > = \frac{1}{\sqrt{\tilde{n}_{\vec{p}}!}} \left(\sqrt{\frac{(2\pi)^3}{V}} \; \hat{\tilde{a}}_{\vec{p}}^\dagger \right)^{\tilde{n}_{\vec{p}}} |\tilde{0} >_{(\vec{p})} \tag{28}$$

$$\sqrt{\frac{(2\pi)^3}{V}} \; \hat{\tilde{a}}_{\vec{p}}^\dagger \, |\tilde{n}_{\vec{p}} > = \sqrt{\tilde{n}_{\vec{p}} + 1} \, |\tilde{n}_{\vec{p}} + 1 > \tag{29}$$

$$\sqrt{\frac{(2\pi)^3}{V}} \; \hat{\tilde{a}}_{\vec{p}} \, |\tilde{n}_{\vec{p}} > = \sqrt{\tilde{n}_{\vec{p}}} \, |\tilde{n}_{\vec{p}} - 1 > \tag{30}$$

$$< \tilde{n}_{\vec{p}} | \tilde{n}'_{\vec{p}'} > = \left(\frac{(2\pi)^3}{V} \right) \delta^{\tilde{n}_{(\vec{p})} \tilde{n}'_{(\vec{p}')}} \delta^{(3)}(\vec{p} - \vec{p}'), \; < \tilde{0} | \tilde{0} > = 1 \,. \tag{31}$$

Furthermore, the eigenstates and eigenvalues of the normal ordered Hamiltonian part, $: \hat{H}(\hat{\tilde{a}}) := \int d^3p \; E_p \hat{\tilde{\mathcal{N}}}_{\vec{p}}$ ['half' of Eq.([26])], read as,

$$: \hat{H}(\hat{\tilde{a}}) : \; (|\tilde{n}_{\vec{p}} > \otimes |\tilde{n}_{\vec{p}\,'} > \otimes ...) = \left(\sum_{\vec{k} = \vec{p}, \vec{p}\,', ...} \tilde{n}_{\vec{k}} \, E_{\vec{k}} \right) (|\tilde{n}_{\vec{p}} > \otimes |\tilde{n}_{\vec{p}\,'} > \otimes ...) \,. \tag{32}$$

• As a consequence of the two above bullet points, the eigenstates and eigenvalues of the normal ordered Hamiltonian of Eq.([26]) read as,

$$: \hat{H} : \ (|n_{\vec{p}} > \otimes |n_{\vec{p}\,'} > \otimes ...) \otimes (|\tilde{n}_{\vec{p}} > \otimes |\tilde{n}_{\vec{p}\,'} > \otimes ...) \tag{33}$$
$$= \left(\sum_{\vec{k}=\vec{p},\vec{p}\,',...} n_{\vec{k}} E_k + \sum_{\vec{\tilde{k}}=\vec{p},\vec{p}\,',...} \tilde{n}_{\vec{\tilde{k}}} E_{\tilde{k}} \right) (|n_{\vec{p}} > \otimes |n_{\vec{p}\,'} > \otimes ...) \otimes (|\tilde{n}_{\vec{p}} > \otimes |\tilde{n}_{\vec{p}\,'} > \otimes ...) \, .$$

Within this free framework, one could have equivalently chosen $\hat{\phi}_1(\hat{a}_{1p}), \hat{\phi}_2(\hat{a}_{2p})$ as the two species of particles. Nevertheless, the presence of interactions may point towards the necessity of using $\hat{\phi}^{(\dagger)}(\hat{a}_p, \hat{\tilde{a}}_p)$, as we are going to discuss in the next subsection.

4.5 Noether's theorem

$\hat{\mathcal{L}}$ in Eq.([23]) is **invariant** under a global 'gauge' transformation.

Remark : The following discussion remains true for the real gauge transformations of nature [local parameter $\theta = \theta(x^\nu)$] arising when fully including the interactions in the Lagrangian.

Global transformations here:

$$\begin{cases} \hat{\phi}(x^\nu) \to e^{-iq\theta} \hat{\phi}(x^\nu) \\ \hat{\phi}^\dagger(x^\nu) \to e^{iq\theta} \hat{\phi}^\dagger(x^\nu) \end{cases}$$

$$\tau \begin{cases} x^\mu \to x'^\mu = x^\mu \\ \hat{\phi}(x^\nu) \to \hat{\phi}'(x'^\nu) \simeq \hat{\phi}(x^\nu) \underbrace{-iq\theta\hat{\phi}(x^\nu)}_{\delta\hat{\phi}} \quad \text{and} \quad \delta\hat{\phi}^\dagger = iq\theta\hat{\phi}^\dagger \end{cases}$$

Then the Noether's theorem predicts the current **conservation** relation (*local conservation*), $\partial_\mu \hat{J}^\mu = 0$, where,

$$\begin{aligned} \hat{J}^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\hat{\phi} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \delta\hat{\phi}^\dagger - \hat{T}^{\mu\nu} \delta x_\nu \\ &= \partial^\mu \hat{\phi}^\dagger (-iq\theta\hat{\phi}) + \partial^\mu \hat{\phi} (iq\theta\hat{\phi}^\dagger) \\ &= iq\theta[-(\partial^\mu \hat{\phi}^\dagger)\hat{\phi} + (\partial^\mu \hat{\phi})\hat{\phi}^\dagger] \, . \end{aligned}$$

J^μ is thus defined up to a factor and a constant so the q-charge as well: only relative charges, among several fields, do matter and only interactions allow their determinations.

Even classically: the electric charge is 'absolute' by convention in the Maxwell's equations since a q-redefinition can be done with

$$\left(\begin{array}{l} \vec{B} \\ \vec{E} \end{array} \right. \quad \text{unit redefinition e.g. in } \vec{\nabla}.\vec{E} = \frac{q\rho}{\epsilon_o} \ .$$

Now integrating, over the whole space, the local conservation relation, and using the Gauss's theorem, one finds directly the *global conservation* equation:

$$\begin{aligned} \frac{d\hat{Q}}{dt} = 0 \ \text{ with } \ \hat{Q} &= \int d^3x \ \hat{J}^0 \\ &= iq \int d^3x \ [(\partial^0\hat{\phi})\hat{\phi}^\dagger - (\partial^0\hat{\phi}^\dagger)\hat{\phi}] \\ &= \dots \\ &= q \int d^3p \ [\underbrace{\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})}_{\hat{\mathcal{N}}_{\vec{p}}} - \underbrace{\hat{\tilde{a}}^\dagger(\vec{p})\hat{\tilde{a}}(\vec{p})}_{\hat{\tilde{\mathcal{N}}}_{\vec{p}}}] \\ &= q\hat{\mathcal{N}} + (-q)\hat{\tilde{\mathcal{N}}} \ . \end{aligned}$$

Conclusion n'1:

$$\left\{ \begin{array}{l} \tilde{a} \text{ annihilates an anti-particle} \\ \tilde{a}^\dagger \text{ creates an anti-particle} \end{array} \right.$$

$\Rightarrow$ $\hat{\phi}(x^\nu)$ annihilates a particle / creates an anti-particle, i.e. decreases the global Q-charge by an amount 'q'. Similarly $\hat{\phi}^\dagger(x^\nu)$ increases Q by 'q'.

Conclusion n'2: In Quantum ElectroDynamics (QED), $\hat{\phi}, \hat{\phi}^\dagger(x^\nu)$ field operators rather imposed as being under the real form $\hat{\phi}_1, \hat{\phi}_2(x^\nu)$ would not allow a consistent charge description since one would get instead problematic mixed terms as,

$$\hat{Q} = iq \int d^3p \ [\hat{a}_1^\dagger(\vec{p})\hat{a}_2(\vec{p}) - \hat{a}_2^\dagger(\vec{p})\hat{a}_1(\vec{p})] \ .$$

5 Propagators

5.1 Solving the generic K.-G. equation

Stucking in this part to the relativistic quantum framework (classical field theory), the K.-G. equation with a source (interaction) term is,

$$(\Box + m^2)\phi(x^\nu) = S(x^\mu)$$

for ϕ being a real [like in Eq.([8])-([9])] or complex [like in Eq.([23])] scalar field. How to solve this equation ? For this purpose, let us introduce the so-called Green's function, defined through,

$$(\Box_x + m^2)G(x_\mu - x'_\mu) \;\hat{=}\; -\,\delta^{(4)}(x_\mu - x'_\mu)\ . \qquad ([28])$$

Remark : The Green's function also permits to solve the Maxwell's equations (relativistic and quantum) taking the similar form, $0 = \partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu$, in a specific gauge for the case free of charge distributions.

Indeed if Eq.([28]) is solvable, i.e. if a solution exists for G, then one knows the solution of the generic K.-G. equation for any integrable $S(x^\alpha)$:

$$\phi_{\text{sol.}}(x_\mu) = \phi_0(x_\mu) - \int d^4x'\ G(x_\mu - x'_\mu)\ S(x'_\mu) \qquad ([29])$$

where ϕ_0 is the solution of the free/original K.-G. equation [for $S(x_\mu) = 0$]. Check :

$$(\Box_{(x)} + m^2)\phi_{\text{sol.}}(x^\nu) = \underbrace{(\Box_x + m^2)\phi_0(x_\mu)}_{=0} - \int d^4x'\, \underbrace{(\Box_x + m^2)G(x_\mu - x'_\mu)}_{-\delta^{(4)}(x_\mu - x'_\mu)}\, S(x'_\mu) = S(x_\mu)\ .$$

So let us find out $G(x_\mu - x'_\mu)$ from Eq.([28]), through the Fourier transformation:

$$G(x^\mu - x'^\mu) = \int \frac{d^4p}{(2\pi)^4}\ e^{-i\underline{p}.(x-x')}\ G(p^\mu) \qquad ([29]\text{bis})$$

with independent p^0 and E_p (or $\underline{p}.\underline{p} = (p^0)^2 - (\vec{p})^2$ and m^2) consistently with the potential presence of interactions (possible quantum fluctuations). Then Eq.([28]) gives the following information,

$$-\delta^{(4)}(x_\mu - x'_\mu) = \int \frac{d^4p}{(2\pi)^4} G(p^\mu)(\underbrace{\Box_x}_{(*)} + m^2)\ \underbrace{e^{-i\underline{p}.(x-x')}} = \underbrace{\int \frac{d^4p}{(2\pi)^4} e^{-i\underline{p}.(x-x')}}_{\delta^{(4)}(x_\mu - x'_\mu)}(-\underline{p}^2 + m^2)G(p^\mu)$$

$$\Leftrightarrow\ \boxed{G(p^\mu) = \frac{1}{\underline{p}^2 - m^2}} = \frac{1}{(p^0)^2 - \vec{p}^2 - m^2} = \frac{1}{(p^0)^2 - E_p^2}\ .$$

(*) $\partial^\mu \partial_\mu (-ip_\rho x^\rho)\, e^{-ip_{\cdot}(x-x')} = (-ip_\mu)\partial^\mu(-ip^\rho x_\rho)\, e^{-ip_{\cdot}(x-x')} = -p^2\, e^{-ip_{\cdot}(x-x')}$
This propagator will appear in the Feynman rules. It remains to Fourier transform $G(p^\alpha)$ to get $G(x_\alpha - x'_\alpha) \Rightarrow$ problem to know how dp_0-integrate over the 2 poles, at $p_0 = \pm E_p$, of the integrant (in general, integrability around a pole: question of the convergence rapidity).

———

One has to introduce, $F(p_\alpha) \equiv \frac{1}{(p^0)^2-(E_p-i\epsilon)^2}$, and then take the (real) limit $\epsilon \to 0$. This is not necessary in quantum field theory where $\epsilon \sim \Gamma$, $\Gamma\tau = \hbar$ (dimension $[E]T$). This regularisation is called the Feynman prescription : introduce a complex number via defining, $G(p_\alpha) = \lim_{\epsilon\to 0} F(p_\alpha)$, to reject the two poles outside the real axis. One has first to separate the 2 poles:

$$F(p_\alpha) = \frac{1}{2[E_p - i\epsilon]}\left[\frac{p_0 + (E_p - i\epsilon)}{p_0 + (E_p - i\epsilon)}\,\frac{1}{p^0 - (E_p - i\epsilon)} - \frac{p^0 - (E_p - i\epsilon)}{p^0 - (E_p - i\epsilon)}\,\frac{1}{p^0 + (E_p - i\epsilon)}\right]$$

$$G(x_\alpha - x'_\alpha) = \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}.(\vec{x}-\vec{x}')}}{2E_p} \int \frac{dp^0}{(2\pi)} e^{-ip^0(t-t')} \times \lim_{\epsilon\to 0}\left[\frac{1}{p^0 - E_p + i\epsilon} - \frac{1}{p^0 + E_p - i\epsilon}\right].$$

Mathematical result from the residue theorem application [2] (clockwise, that is negative orientation, along a rectangle path around the pole with a side corresponding to the real axis):

$$0+0+0+\int_{-\infty}^{+\infty} dX\,\frac{e^{-iXt}}{X+i\epsilon} = -2i\pi\,\theta(t)e^{-i(-i\epsilon)t} \quad\Rightarrow\quad \lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} dX\,\frac{e^{-iXt}}{X+i\epsilon} = -i2\pi\,\theta(t)\,.$$

Let us use this formal result for our purpose: the first term (second integration) of the $G(x_\alpha - x'_\alpha)$ expression reads as,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} \frac{dp_0}{2\pi}\,\frac{e^{-ip_0(t-t')}}{p^0 - E_p + i\epsilon} &= \lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} \frac{dp'_0}{2\pi}\,\frac{e^{-ip'_0(t-t')}}{p'_0 + i\epsilon}e^{-iE_p(t-t')} \\ &= \frac{e^{-iE_p(t-t')}}{2\pi}[-2i\pi\,\theta(t-t')]\,.\end{aligned}$$

The second term (second integration) of the $G(x_\alpha - x'_\alpha)$ expression reads as,

[2] $\epsilon < 0$ possible and equivalent but changes the analytical calculation.

$$\lim_{\epsilon\to 0}\int \frac{dp_0}{2\pi}\frac{-e^{-ip_0(t-t')}}{p^0+E_p-i\epsilon} \;=\; \lim_{\epsilon\to 0}(-1)\int_{-\infty}^{+\infty}\frac{(-dp_0')}{2\pi}\;\frac{-e^{-ip_0'(t'-t)}}{-(p_0'+i\epsilon)}e^{-iE_p(t'-t)}$$
$$= \frac{e^{-iE_p(t'-t)}}{2\pi}[-2i\pi\,\theta(t'-t)]\,.$$

Therefore,

$$G(x_\alpha - x'_\alpha) = -i\left[\int_{-\infty}^{+\infty}\frac{d^3p}{(2\pi)^3}e^{-iE_p(t-t')}\theta(t-t')e^{i\vec{p}.(\vec{x}-\vec{x}')} + \int_{-\infty}^{+\infty}\frac{-d^3(-p)}{(2\pi)^3}e^{-iE_p(t'-t)}\theta(t'-t)e^{i(-\vec{p}).(-\vec{x}+\vec{x}')}\right]\frac{1}{2E_p}$$

$$\boxed{G(x_\alpha - x'_\alpha) = -i\int_{-\infty}^{+\infty}\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left[\theta(t-t')\;e^{-ip_{\cdot}(x-x')} + \theta(t'-t)\;e^{-ip_{\cdot}(x'-x)}\right]} \qquad ([30])$$

with formally p^0 re-introduced and taken to be E_p (a possibility within the free case). Notice that within the natural unit system, $[G(x)] = [E]^2$ [consistently with Eq.([28])].

5.2 Introducing the time-ordering

Let us now connect $G(x_\alpha - x'_\alpha)$ to the quantised scalar field operator $\hat{\phi}(x^\mu)$.

[A] For the real scalar field, using Eq.([16]) and Eq.(18),

$$\hat{\phi}(x^\mu)|0> = \int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\;\hat{a}^\dagger(\vec{p})\;e^{ix_{\cdot}p}|0> = \int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\;e^{ix_{\cdot}p}|1>_{(\vec{p})}\;\sqrt{\frac{V}{(2\pi)^3}}\,.$$

Taking now the Hermitian conjugate sequence of this equality,

$$<0|\hat{\phi}^\dagger(x'_\mu) = \int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}\;e^{-ip'_{\cdot}x'}\;<1|_{(\vec{p}')}\sqrt{\frac{V}{(2\pi)^3}}$$ recalling that here, $\hat{\phi}^\dagger(x^\mu) = \hat{\phi}(x^\mu)$,

we get,

$$<0|\hat{\phi}^{(\dagger)}(x'_\mu)\hat{\phi}(x_\mu)|0> = \int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\int\frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}\;e^{i(p_{\cdot}x-p'_{\cdot}x')}\underbrace{{}_{(\vec{p}')}<1|1>_{(\vec{p})}\frac{V}{(2\pi)^3}}_{\delta^{(3)}(\vec{p}-\vec{p}')}$$
$$= \int\frac{d^3p}{(2\pi)^3 2E_p}\;e^{ip_{\cdot}(x-x')}\;.$$

based on Eq.(21) and the equality $p^0 = E_p$ (free field expression origin). Using Eq.([30]), one can thus link the quantised field with the Green function,

$$iG(x_\alpha - x'_\alpha) = \theta(t-t') <0|\hat{\phi}(x_\mu)\hat{\phi}(x'_\mu)|0> + \theta(t'-t) <0|\hat{\phi}(x'_\mu)\hat{\phi}(x_\mu)|0>\;. \qquad (34)$$

We recover the dimensional analysis $[G(x)] = [E]^2$. Let us rewrite this equation as,

$$\boxed{iG(x_\alpha - x'_\alpha) =< 0|\tau[\hat{\phi}(x_\mu)\hat{\phi}(x'_\mu)]|0 >} \qquad ([31])$$

with $\tau[\hat{\phi}_1(x^\nu)\hat{\phi}_2(x'^\nu)] \hat{=} \begin{cases} \hat{\phi}_1(x^\nu)\hat{\phi}_2(x'^\nu) & \text{if } t > t' \\ \hat{\phi}_2(x'^\nu)\hat{\phi}_1(x^\nu) & \text{if } t' > t \end{cases}$

Indeed, e.g. for $t > t'$, Eq.(34) leads to:
$iG(x_\alpha - x'_\alpha) = 1\times < 0|\hat{\phi}(x_\mu)\hat{\phi}(x'_\mu)|0 > +0 \hat{=} < 0|\tau[\hat{\phi}(x_\mu)\hat{\phi}(x'_\mu)]|0 >$.

Given this "time-ordering" definition, Eq.([31]) can be recast into

$$iG(x_\alpha - x'_\alpha) = < 0|\tau[\hat{\phi}(x'_\mu)\hat{\phi}(x_\mu)]|0 > \; .$$

Physical interpretation (still for the example case $t > t'$):

- $\hat{\phi}(x'_\mu)|0 > \sim \int d^3p \; |1 >_{(\vec{p})}$ (up to coefficients) so creation of a particle at time t' of species $\hat{\phi}$,
- $\hat{\phi}(x_\mu) \ni \int d^3p' \hat{a}(\vec{p'})$, leading to $< 0|0 > \neq 0$ so annihilation of a particle of species $\hat{\phi}$ at time t.

So the matrix element $\overbrace{< 0|\tau[\hat{\phi}(x_\mu)\hat{\phi}(x'_\mu)]|0 >}^{iG(x_\mu - x'_\mu)}$ can describe the amplitude of a particle being created at t', propagating in space-time and being annihilated at posterior moment t. It will turn out to enter the calculation of amplitudes for reactions (involving interactions: the vertices). This can be expected intuitively, as after all, in nature, particles exclusively propagate freely in space-time or interact with (/ transform to) other particles.

Remark : This Green function meaning differs slightly from the interpretation in relativistic quantum theory: Eq.([29]) (where it drives radiation propagation).

[B] For the complex scalar field, using Eq.(26) and Eq.([30]), one can show that (possible homework),

$$\boxed{iG(x_\mu - x'_\mu) =< 0|\tau[\hat{\phi}(x^\nu)\hat{\phi}^\dagger(x'^\nu)]|0 >} \qquad ([32])$$

Regarding the physical interpretation, notice that the field product $\hat{\phi}(x^\nu)\hat{\phi}^\dagger(x'^\nu)$, respectively $\hat{\phi}^\dagger(x'^\nu)\hat{\phi}(x^\nu)$, involves in particular the operator product $\hat{a}_p\hat{a}^\dagger_p$, respectively $\hat{\tilde{a}}_p\hat{\tilde{a}}^\dagger_p$. Finally, there is thus a unique Green function / propagator definition for both the real and complex scalar fields.

6 Interactions and evolution operator

6.1 Structure of interactions

So far, only free particles studied (in a boring world) with quadratic terms in $\mathcal{L}$ [mass or kinetic terms]. Notice the possibility, not studied here, of a term like $m\phi_2^\dagger\phi_1$ 'mixing' 2 particles so that a state rotation is then needed to obtain 2 independent states.

What do the interaction terms could look like in general? $\begin{cases} \to \text{Lorentz invariant} \\ \to \mathcal{L} \text{ Hermitian (up to total derivatives)} \end{cases}$

• For a real scalar field $\hat{\phi}$, the Lagrangian of interactions reads generically as,

$$\hat{\mathcal{L}}_I = -g\hat{\phi}^3 - \lambda\hat{\phi}^4 - \sum_{k=5}^{N} \frac{\lambda_k \hat{\phi}^k}{(k!)} \quad \begin{cases} \to \hat{\phi}(x) \text{ Lorentz invariant by itself} \\ \to \hat{\phi} = \hat{\phi}^\dagger \end{cases}$$

recalling that,

$$\hat{\mathcal{L}} = \hat{\pi}_A \dot{\hat{\phi}}_A - \hat{\mathcal{H}} = \hat{\pi}_A \dot{\hat{\phi}}_A - (\hat{\mathcal{H}}_{kin} + \hat{\mathcal{V}}) \ .$$

The coupling constants determine the strength of the interactions : $[g] = E$, $[\lambda] = 1,\ [\lambda_k] = [E]^{-1}, [E]^{-2}, ...$

• For a complex scalar field $\hat{\phi}$, the Lagrangian of interactions may contain for example,

$$\hat{\mathcal{L}}_I \ni \ \text{same terms or e.g. } \hat{\phi}^\dagger\hat{\phi}^2 \text{ (mixed)} \quad \begin{cases} \to \hat{\phi}^\dagger \text{ also invariant (another field)} \\ \quad \hat{\phi}(x) = \hat{\phi}'(x') \quad \text{so} \quad \hat{\phi}^\dagger(x) = \hat{\phi}'^\dagger(x') \\ \to +H.c. \end{cases} \ .$$

The nature and structure of the interaction terms define the specific theory (possibly hypothetical).

6.2 Working out the physical states

Quantum states (of a generic system) known

$\Rightarrow \begin{cases} \text{eigenvalues of observables known, } \hat{\mathcal{O}}|\alpha >= \alpha|\alpha > \\ \text{transition amplitudes known, } P = |<\beta|\alpha>|^2 \end{cases}$

Now the eigenstates of the Hamiltonian $\hat{H}$ (main observable and related to $\hat{\mathcal{L}}$) are generically hard to find out (in QFT too). Therefore, one is usually constrained to consider an

Hamiltonian, $\hat{H} = \hat{H}_0 + \hat{H}_I \ni$ interactions typically small, in order to develop perturbation approaches. Here, $\hat{H}_0$ represents the free Hamiltonian. As usually in perturbation theory, $\hat{H}_0$ and $\hat{H}_I$ belong to the same $\mathbb{H}$-space whose possible Ortho-Normal-Basis is noted as the generic set [involving the known $\hat{H}_0$ eigenstates worked out in Eq.(33)]:

$$\{...|n_{\vec{p}} > |n'_{\vec{p}+\vec{\delta p}} > ...\} = \{|E >\} .$$

Hence, considering the relevant Hamiltonian operators – along the time axis – to be:

$$\hat{H}_0 \ \rightarrow \ \hat{H}_0 + \hat{H}_I(t) \text{ [as used here or possibly adiabatic } \hat{H}_I \text{ perturbation]} \ \rightarrow \ \hat{H}_0$$

the initial [produced] and final [detected] multi-particle states (before and after the studied reaction among particles) are defined as free states and in turn as classical states. The Schrödinger equation in the studied Fock space, providing the quantum state evolution, is

$$\boxed{i(\hbar)\frac{d}{dt}|\Psi(t) >= (\hat{H}_0 + \hat{H}_I)|\Psi(t) >} \qquad ([33])$$

and for a free state,

$$i(\hbar)\frac{d}{dt}|\Psi_0(t) >= \hat{H}_0|\Psi_0(t) > \qquad ([34])$$

Let us introduce the evolution operator $\hat{U}_0(t)$ relating '$-\infty$' to 't' for the free state:

$$|\Psi_0(t) >= \hat{U}_0(t)|\Psi_0(-\infty) > \hat{=} \hat{U}_0(t)|i > \qquad ([35])$$

Let us then introduce the evolution operator $\hat{U}(t)$ relating 'free' states with 'coupled' states:

$$|\Psi(t) >= \hat{U}_0(t)\hat{U}(t)\hat{U}_0^\dagger(t)|\Psi_0(t) > \quad (\forall t) \qquad ([36])$$

Eq.([35]) and Eq.([36]) synthesize into the relation,

$$\boxed{|\Psi(t) >= \hat{U}_0(t)\hat{U}(t)|i >} \quad (|i > \hat{=} |\Psi_0(-\infty) >) \qquad ([37])$$

using the unitarity property, $\hat{U}_0^\dagger(t) = \hat{U}_0^{-1}(t)$. The unitarity for both operators, $\hat{U}(t)$ and $\hat{U}_0(t)$ (also demonstrated in Tutorials 3, Exercise 3) is allowing to conserve the quantum state normalisations:

$$< \Psi_0(t)|\Psi_0(t) >=< i|\hat{U}_0^\dagger\hat{U}_0|i > \text{ and } < \Psi(t)|\Psi(t) >= (< \Psi_0(t)|\hat{U}_0\hat{U}^\dagger\hat{U}_0^\dagger)(U_0\hat{U}\hat{U}_0^\dagger|\Psi_0(t) >) . \qquad (35)$$

Therefore, let us work out $\hat{U}_0(t)$ and $\hat{U}(t)$ to obtain the quantum states.

• Let us first work out $\hat{U}_0(t)$. Eq.([34]) and Eq.([35]) lead to,

$$i\frac{d}{dt}\hat{U}_0(t)|i >= \hat{H}_0\hat{U}_0(t)|i >= \hat{U}_0(t)\underbrace{\hat{U}_0^\dagger(t)\hat{H}_0\hat{U}_0(t)}_{\hat{H}_0(t)}|i > \quad (\forall|i >\neq 0) \qquad ([38]\text{bis})$$

whose solution reads as (see Tutorials 3, Exercise 3),

$$\boxed{\hat{U}_0(t) = e^{-i\hat{H}_0 t}} \qquad ([38])$$

• We now focus on $\hat{U}(t)$. The Eq.([33]) and Eq.([37]) allow us to write,

$$i\frac{d}{dt}(\hat{U}_0(t)\hat{U}(t)|i>) = (\hat{H}_0 + \hat{H}_I)\hat{U}_0(t)\hat{U}(t)|i> \quad (\forall |i> \neq 0)$$

$$i\left(\hat{U}_0'(t)\hat{U}(t) + \hat{U}_0(t)\hat{U}'(t)\right)|i> =$$

$$\left(\hat{H}_0\hat{U}_0(t)\hat{U}(t) + i\hat{U}_0(t)\hat{U}'(t)\right)|i> =$$

$$i\frac{d\hat{U}(t)}{dt} = \underbrace{\hat{U}_0^\dagger(t)\hat{H}_I\hat{U}_0(t)}_{\hat{H}_I(t)}\hat{U}(t) \qquad ([39])$$

Eq.([38]bis) was used in the last line. Interpretation of $\hat{H}_I(t) \equiv \hat{U}_0^\dagger(t)\hat{H}_I\hat{U}_0(t)$ (based on Eq.([35])):

$$<\beta|\hat{H}_I(t)|\beta> = <\beta|\hat{U}_0^\dagger(t)\hat{H}_I\hat{U}_0(t)|\beta> = <\beta(t)|\hat{H}_I|\beta(t)>$$

"Heisenberg picture" $[\,|\phi\rangle\,]$ $\leftrightarrow$ "Schrödinger picture" $[\,\phi(x,t)\,]$

where $\hat{U}_0(t)$ is acceptable up to perturbation order corrections (in contrast with the exact evolution operator involved in: $|\Psi(t)> = e^{-i(\hat{H}_0+\hat{H}_I)t}|i>$). Now in order to solve Eq.([39]), let us first integrate it over the following domain,

$\int_{t_0}^{t} dt_1 \text{Eq.([39])}: \quad \hat{\mathrm{U}}(\mathrm{t}) - \hat{\mathrm{U}}(\mathrm{t_0}) = -\mathrm{i}\int_{\mathrm{t_0}}^{\mathrm{t}} \mathrm{dt_1}\hat{\mathrm{H}}_\mathrm{I}(\mathrm{t_1})\hat{\mathrm{U}}(\mathrm{t_1}),$

$t \to -\infty : \hat{H}_I(t) \to 0, \dfrac{d\hat{U}(t)}{dt} \to 0 \Rightarrow \hat{U}(t) \to \mathbb{1}$ as $\hat{U}(t)$ implements the interaction [Eq.([36])].

Hence, for $t_0 \to -\infty$: $\hat{U}(t) = \mathbb{1} - i\int_{-\infty}^{t} dt_1 \hat{H}_I(t_1)\hat{U}(t_1)$.

One can then use an iterative approach,

$$
\begin{aligned}
\hat{U}(t) &= \mathbb{1} - i\int_{-\infty}^{t} dt_1\, \hat{H}_I(t_1)[\mathbb{1} - i\int_{-\infty}^{t_1} dt_2\, \hat{H}_I(t_2)\hat{U}(t_2)] \\
&= \mathbb{1} - i\int_{-\infty}^{t} dt_1\, \hat{H}_I(t_1) + (-i)^2\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, \hat{H}_I(t_1)\hat{H}_I(t_2)\hat{U}(t_2) \\
&= \mathbb{1} - i\int_{-\infty}^{t} dt_1\, \hat{H}_I(t_1) + (-i)^2\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, \hat{H}_I(t_1)\hat{H}_I(t_2) \\
&\quad + (-i)^3\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3\, \hat{H}_I(t_1)\hat{H}_I(t_2)\hat{H}_I(t_3)\hat{U}(t_3) \\
&= ... \\
&= \mathbb{1} + \sum_{n=1}^{\infty}(-i)^n \int_{-\infty}^{(t_0\hat{=})\,t} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n\, \hat{H}_I(t_1)...\hat{H}_I(t_n)\,.
\end{aligned}
$$

Let us rewrite the third term ($n = 2$),

$$
\begin{aligned}
\frac{(-i)^2}{2!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2\; \tau[\hat{H}_I(t_1)\hat{H}_I(t_2)] = \frac{(-i)^2}{2!}\Bigg[\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\; \theta(t_1 - t_2)\hat{H}_I(t_1)\hat{H}_I(t_2) \\
+ \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1\; \theta(t_2 - t_1)\hat{H}_I(t_2)\hat{H}_I(t_1)\Bigg]\,.
\end{aligned}
$$

Indeed, renaming, $t_1 \leftrightarrow t_2$, in the last term we get,

$$
\begin{aligned}
\frac{(-i)^2}{2!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2\; \tau[\hat{H}_I(t_1)\hat{H}_I(t_2)] = \frac{(-i)^2}{2!}\Bigg[\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\; \theta(t_1 - t_2)\hat{H}_I(t_1)\hat{H}_I(t_2) \\
+ \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\; \theta(t_1 - t_2)\hat{H}_I(t_1)\hat{H}_I(t_2)\Bigg]
\end{aligned}
$$

which gives, encoding the θ step function informations only into the integration region boundaries and identifying the two terms,

$$
\frac{(-i)^2}{2!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2\; \tau[\hat{H}_I(t_1)\hat{H}_I(t_2)] \;=\; (-i)^2\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\; \hat{H}_I(t_1)\hat{H}_I(t_2)\,.
$$

Therefore $\hat{U}(t)$ can be rewritten in a more compact form (convenient for the following), when generalising to higher terms as,

$$
\boxed{\hat{U}(t) = \mathbb{1} + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \ldots \int_{-\infty}^{t} dt_n\; \tau\Big[\hat{H}_I(t_1)\hat{H}_I(t_2)...\hat{H}_I(t_n)\Big]} \qquad ([40])
$$

$\hat{U}(t)$ is thus solved and exhibits a possible perturbative treatment: $\hat{H}_I(t)$ having typically small eigenvalues, the first terms of this series are expected to be dominant and can thus be exclusively considered – in a first approximation.

General warning: in Quantum Field Theory, such series are in general not convergent (even for 'small' $\hat{H}_I$'s) which questions the validity of the QFT formalism in general and its quantitative predictions as well (same comment for non-canonical quantisations). This represents a non-trivial open question but the experimental tests at accelerators on particle reaction amplitudes have been passed brillantly so far so that phenomenologically the theory is acceptable in its present form.

6.3 The $\hat{S}$ matrix

$$\hat{S} \hat{=} \lim_{t \to \infty} \hat{U}(t) \qquad ([41])$$

This definition implies the unitarity of $\hat{S}$ as $\hat{U}(t)$ [related to probabilities via in particular Eq.(35)]: $\hat{S}^\dagger \hat{S} = \mathbb{1}$. The evolution operator $\hat{U}(t)$, and in turn $\hat{S}$, can be rewritten as,

$$\boxed{\hat{S} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d^4x_1 \ldots \int_{-\infty}^{+\infty} d^4x_n \ \tau \Big[\hat{\mathcal{H}}_I(x_1^\mu) \ldots \hat{\mathcal{H}}_I(x_n^\mu) \Big]} \qquad ([42])$$

or even in a neater form (rendering also the normal order explicit):

$$\hat{S} = \tau \Big[\ exp \Big(-i \int d^4x : \hat{\mathcal{H}}_I(x) : \Big) \ \Big]. \qquad (36)$$

What is the motivation for considering the $\hat{S}$-matrix element?

The interesting quantity is the amplitude of transition : $< f|...|i >$. When interactions have ceased, the final state $|f>$ can also be taken among relevant $\hat{H}_0$'s eigenstates (quantum formalism consistency $\Rightarrow$ identical time origin):

$$< f|\hat{S}|i >= \lim_{t \to \infty} < f|\hat{U}(t)|i >= \lim_{t \to \infty} \underbrace{< f|\hat{U}_0^\dagger(t)}_{<f(t)|} \underbrace{\hat{U}_0(t)\hat{U}(t)|i >}_{|\Psi(t)>} \equiv \text{Amplitude of reaction.} \qquad ([42]\text{bis})$$

The quantum approach is as announced : $\hat{U}(t) \Rightarrow |\Psi(t) > \Rightarrow$ Amplitudes.

7 Reaction amplitudes

7.1 Wick's theorem

7.1.1 Wick contraction

Real field:

$$\underbracket{\hat{\phi}(x)\hat{\phi}}(x') \hat{=} \underbrace{<0|\tau[\hat{\phi}(x_\mu)\hat{\phi}(x'_\mu)]|0>}_{iG(x_\mu - x'_\mu) \text{ via Eq.([31])}} \mathbb{1} = \underbracket{\hat{\phi}(x')\hat{\phi}}(x) \qquad (37)$$

$$\underbracket{\hat{\phi}(x)\hat{\phi}'}(x') = <0| \left[\theta(t-t')\hat{\phi}(x)\hat{\phi}'(x') + \theta(t'-t)\hat{\phi}'(x')\hat{\phi}(x)\right] |0> \mathbb{1} = 0 \qquad (38)$$

since e.g. for the first term, $\hat{\phi} \sim \hat{a} + \hat{a}^\dagger$, $\hat{\phi}' \sim \hat{a}' + \hat{a}'^\dagger$ and $(<1_p|<0'|)(|0>|1'_{p'}>) =$ $<1_p|0> \times <0'|1'_{p'}> = 0 \times 0 = 0$ from Eq.(21).

Complex field:

Using the result of the Tutorials 3, Exercise 2 (in the second line below), we now define,

$$\underbracket{\hat{\phi}(x)\hat{\phi}^\dagger}(x') \hat{=} \underbrace{<0|\tau[\hat{\phi}(x)\hat{\phi}^\dagger(x')]|0>}_{iG(x_\mu - x'_\mu) \text{ via Eq.([32])}} \mathbb{1} = \hat{\phi}^\dagger\underbracket{(x')\hat{\phi}}(x)$$

$$= <0|\tau[\hat{\phi}(x')\hat{\phi}^\dagger(x)]|0> \mathbb{1} = \underbracket{\hat{\phi}(x')\hat{\phi}^\dagger}(x) = \hat{\phi}^\dagger\underbracket{(x)\hat{\phi}}(x')$$

As for the real field : $\hat{\phi}^{(\dagger)}\underbracket{(x)\hat{\phi}'^{(\dagger)}}(x') = 0\,.$

$$\underbracket{\hat{\phi}(x)\hat{\phi}}(x') \hat{=} <0|\tau[\hat{\phi}(x)\hat{\phi}(x')]|0> \mathbb{1} = 0 = \underbracket{\hat{\phi}(x')\hat{\phi}}(x)$$

since e.g. for the first term, $\hat{\phi} \sim \hat{a} + \hat{\tilde{a}}^\dagger$ and $(<1_p|<\tilde{0}|)(|0>|\tilde{1}_{p'}>) =$ $<1_p|0> \times <\tilde{0}|\tilde{1}_{p'}> = 0 \times 0 = 0$ from Eq.(21)-(31).

Similarly, $\hat{\phi}^\dagger\underbracket{(x)\hat{\phi}^\dagger}(x') \hat{=} <0|\tau[\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(x')]|0> \mathbb{1} = 0 = \hat{\phi}^\dagger\underbracket{(x')\hat{\phi}^\dagger}(x)$.

7.1.2 Order properties

A regularisation method leads (see Ref.[Lahiri & Pal]) to the formula [needed for Eq.([42])-(36)]:

$$\boxed{\begin{array}{l}\tau[:\hat{A}(x_1)\hat{B}(x_1)....::\hat{A}(x_2)\hat{B}(x_2)...:\cdots:\hat{A}(x_n)\hat{B}(x_n)...:] \\ \quad = \tau[\hat{A}\hat{B}...(x_1)\ \hat{A}\hat{B}...(x_2)\ \ldots\ \hat{A}\hat{B}...(x_n)]_{\text{e.}\not{t}\text{.c.}}\end{array}} \qquad ([44])$$

where e.$\not{t}$.c. stands for 'no equal-time (Wick) contraction'.

Then one can relate the time order and normal order of a field product through the Wick contraction (see Tutorials 3, Exercise 4):

$$\boxed{\tau[\hat{\phi}_1(x)\hat{\phi}_2(x')] \;=\; :\hat{\phi}_1(x)\hat{\phi}_2(x'): \;+\; \underbrace{\hat{\phi}_1(x)\hat{\phi}_2(x')}} \qquad ([43])$$

Case $\hat{\phi}_1 \neq \hat{\phi}_2$ trivial: $(\tau)[\hat{\phi}_1(x)\hat{\phi}_2(x')] = (:)\hat{\phi}_1(x)(:)\ (:)\hat{\phi}_2(x')(:) + 0$.

7.1.3 The theorem

Extending Eq.([43]) (demonstration for a unique field in Ref.[Peskin & Schroeder]):

$$\boxed{\begin{aligned}\tau[\hat{\phi}_1(x_1)\hat{\phi}_2(x_2)\ldots\hat{\phi}_n(x_n)] = \;& :\hat{\phi}_1(x_1)\ldots\hat{\phi}_n(x_n): \\ & + :\underbrace{\hat{\phi}_1\hat{\phi}_2}\ \hat{\phi}_3\ldots\hat{\phi}_n: \quad +\text{permutations} \\ & + :\underbrace{\hat{\phi}_1\hat{\phi}_2}\ \underbrace{\hat{\phi}_3\hat{\phi}_4}\ldots\hat{\phi}_n: \quad +\text{permutations} \\ & + \ldots \\ & + \begin{cases} :\underbrace{\hat{\phi}_1\hat{\phi}_2}\ldots\underbrace{\hat{\phi}_{n-1}\hat{\phi}_n}: \quad +\text{permutations (if n even)} \\ :\underbrace{\hat{\phi}_1\hat{\phi}_2}\ldots\underbrace{\hat{\phi}_{n-2}\hat{\phi}_{n-1}}\ \hat{\phi}_n: \quad +\text{permutations (if n odd)} \end{cases}\end{aligned}} \qquad ([45])$$

Case of all $\hat{\phi}_n$'s distinct (from each other): τ needless and only the first term on the right-hand side is non-zero, and without normal order anymore (trivial case).

7.2 A simple decay example

7.2.1 The interaction and reaction

$$\begin{cases} \text{Interaction}: \hat{\mathcal{L}}_I = -g\,\hat{\phi}_2^\dagger(x^\nu)\hat{\phi}_2(x^\nu)\hat{\phi}_1(x^\nu) \\ \text{Free part}: \hat{\mathcal{L}}_0(\phi_1) + \hat{\mathcal{L}}_0(\phi_2) \end{cases}$$

with g real $\Rightarrow\ :\hat{\mathcal{H}}: = :\hat{\mathcal{H}}_0(\hat{a}_1^{(\dagger)}) + \hat{\mathcal{H}}_0(\hat{a}_2^{(\dagger)}, \hat{\tilde{a}}_2^{(\dagger)}) + \hat{\mathcal{H}}_I:$ with $:\hat{\mathcal{H}}_I := g :\hat{\phi}_2^\dagger\hat{\phi}_2\hat{\phi}_1|_{x^\mu}:$ since only the "$-\hat{\mathcal{L}}_I$" part of the new Hamiltonian piece is non-zero [see Eq.(12)].

The perturbation method can be used with the obtained decomposition of the free fields (leading order result).

The reaction considered is the particle P_1 decay into the particle P_2 and its anti-particle P_2^c, with respective 4-momenta defined as,

$$\boxed{P_1(k_\mu) \to P_2(p_\mu) + P_2^c(p'_\mu)}$$

which is a kinematically open channel under the condition that $M_1 > 2m_2$, where M_1 (m_2) is the mass of the particle P_1 (P_2). This kinematics condition arises from the conservation of the (time component of the) 4-momentum [$k^0 = p^0 + p'^0$], the classical nature of the initial/final particles [both to be confirmed later on] and the use (always possible) of the centre-of-mass frame where $\sum_i \vec{p}_i = \vec{0}$ (for $c = 1$) [$\vec{k} = \vec{p} + \vec{p'} = \vec{0}$]:

$$\sqrt{M_1^2} = \sqrt{m_2^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^{(\prime)2}}\,.$$

7.2.2 First order contribution

From Eq.([42]), and using the orthonormalisation conditions (21) as well as (31),

$$A_{(P_1 \text{ decay})} =< f|\hat{S}|i> \simeq < 1_p^2 \tilde{1}_{p'}^2| \Big(\not{1} + (-i) \int_{-\infty}^{+\infty} d^4x \; \tau[\, g : \hat{\phi}_2^\dagger(x^\mu)\hat{\phi}_2(x^\mu)\hat{\phi}_1(x^\mu) :] \Big) |1_k^1 > + \dots$$

$$\downarrow \text{ Eq.([44])}$$

$$\tau[g\hat{\phi}_2^\dagger(x^\mu)\hat{\phi}_2(x^\mu)\hat{\phi}_1(x^\mu)]_{\text{e.}\not{t}\text{.c.}}$$

$$\downarrow \text{ Eq.([45])}$$

$$g : \hat{\phi}_2^\dagger(x^\mu)\hat{\phi}_2(x^\mu)\hat{\phi}_1(x^\mu) : +0 \text{ (as no e.t.c.)}$$

Useful notations:

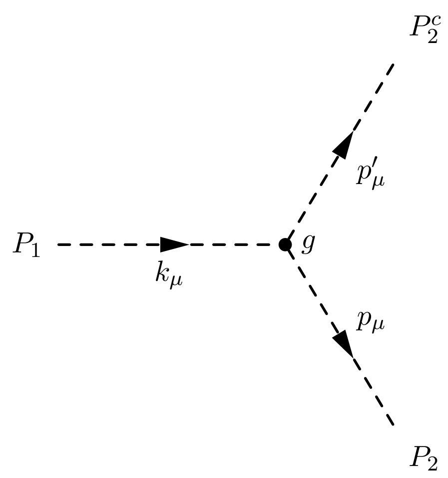
$$\left[\begin{array}{l} \hat{\phi}_2 \,\hat{=} \int \frac{d^3q}{\sqrt{(2\pi)^3 2E_q}} (\hat{a}_{2q} e^{-iq.x} + \hat{\tilde{a}}_{2q}^\dagger e^{+iq.x}) \equiv \hat{\phi}_{2-} + \hat{\phi}_{2+} \\ \hat{\phi}_2^\dagger \,\hat{=} \int \frac{d^3q}{\sqrt{(2\pi)^3 2E_q}} (\hat{\tilde{a}}_{2q} e^{-iq.x} + \hat{a}_{2q}^\dagger e^{+iq.x}) \equiv \hat{\phi}_2^\dagger|_- + \hat{\phi}_2^\dagger|_+ \\ \hat{\phi}_1 \,\hat{=} \int \frac{d^3q}{\sqrt{(2\pi)^3 2E_q}} (\hat{a}_{1q} e^{-iq.x} + \hat{a}_{1q}^\dagger e^{+iq.x}) \equiv \hat{\phi}_{1-} + \hat{\phi}_{1+} \end{array}\right.$$

Using respectively Eq.(19), Eq.(29) and Eq.(20), one finds,

$$A^{(1)} = (-i) \int d^4x \; g \; < 1_p^2 \tilde{1}_{p'}^2|(:) \underbrace{\hat{\phi}_2^\dagger|_+(x^\nu) \; \hat{\phi}_{2+}(x^\nu)}_{\text{creates (anti-)part. } P_2^{(c)}} \; \underbrace{\hat{\phi}_{1-}(x^\nu)}_{\text{annihilates } P_1} (:)|1_k^1 > (|0> ...)\,. \quad ([46])$$

Actions of the fields must be considered on the ket to allow a diagrammatic interpretation (in space-time diagrams), as below.

<u>Feynman diagram:</u>



Such a diagram permits a physical interpretation of the whole reaction amplitude as well as the model Lagrangian.

7.2.3 Next order terms

$$A^{(2)} = \frac{(-ig)^2}{2!} < 1_p^2 1_{p'}^{2c}| \int d^4x_1 \int d^4x_2 \; \tau[: \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1} : : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2} :]|1_k^1 > \; .$$

Using Eq.([44]),

$$A^{(2)} = \frac{(-ig)^2}{2!} < 1_p^2 \tilde{1}_{p'}^2| \int d^4x_1 \int d^4x_2 \; \tau[\hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1} \; \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}]_{\text{e.t/c.}} \, |1_k^1 > \; .$$

Then through Eq.([45]) and the Wick contraction definitions or vanishing properties, we obtain that

$$A^{(2)} = \frac{(-ig)^2}{2!} < 1_p^2 \tilde{1}_{p'}^2| \int d^4x_1 \, d^4x_2 \, \Bigg[: \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1} \; \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2} :$$

$$+ : \underbrace{\hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1} \; \hat{\phi}_2^\dagger} \hat{\phi}_2 \hat{\phi}_1|_{x_2} : + : \hat{\phi}_2^\dagger \underbrace{\hat{\phi}_2 \hat{\phi}_1|_{x_1} \; \hat{\phi}_2^\dagger} \hat{\phi}_2 \hat{\phi}_1|_{x_2} : + : \hat{\phi}_2^\dagger \hat{\phi}_2 \underbrace{\hat{\phi}_1|_{x_1} \; \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1}|_{x_2} :$$

$$+ \, \{\text{terms with more Wick contractions}\} \Bigg] |1_k^1 > \; .$$

Focusing on the first term of this expression and on the Hilbert space of the particle of species 1 (associated to $\hat{\phi}_1$), the found contribution is proportional to,

$$< 0^1| : (\hat{a}_{1q} + \hat{a}_{1q}^\dagger) \, (\hat{a}_{1q'} + \hat{a}_{1q'}^\dagger) : |1_k^1 > = < 0^1|(\hat{a}_{1q}\hat{a}_{1q'} + \hat{a}_{1q}^\dagger \hat{a}_{1q'} + \hat{a}_{1q'}^\dagger \hat{a}_{1q} + \hat{a}_{1q}^\dagger \hat{a}_{1q'}^\dagger)|1_k^1 >$$

$$= < 0^1|(\hat{a}_{1q}\hat{a}_{1q'} + 0 + 0 + 0)|1_k^1 > = < 0^1|\hat{a}_{1q}|0^1 > = \; 0 \, ,$$

where we have used the compact notation for quantum states and Eq.(17). Similarly, the three terms (with a single pair contraction each) on the second line vanish as well since only a power one of the $\hat{\phi}_1$ field can lead to $< 0^1|\hat{a}_{1q}|1^1 > \propto < 0^1|0^1 > \neq 0$ respectively due to Eq.(20) and Eq.(21). As for the terms of the third line, more Wick contractions will not help against those considerations so that they are also vanishing. Hence, we find that $A^{(2)} = 0 = A^{(2n)}$ [n integer].

Let us now move to the next order term:

$$A^{(3)} = \frac{(-ig)^3}{3!} < 1_p^2 \tilde{1}_{p'}^2| \int d^4x_1 \, d^4x_2 \, d^4x_3 \; \tau[: \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1} : : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2} : : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} :]|1_k^1 > \; .$$

Once more, via Eq.([44]), one obtains,

$$A^{(3)} = \frac{(-ig)^3}{3!} < 1_p^2 \tilde{1}_{p'}^2| \int d^4x_1 \, d^4x_2 \, d^4x_3 \; \tau[\hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1} \, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2} \, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3}]_{\text{e.t/c.}} \, |1_k^1 > ,$$

and in turn – keeping in mind that only the field configuration of Eq.([46]) permits a non-vanishing resulting amplitude – Eq.([45]) leads to,

$$
\begin{aligned}
A^{(3)} = (-ig)^3 < 1_p^2 \tilde{1}_{p'}^2 | \int d^4x_1\, d^4x_2\, d^4x_3 \ \Big[&: \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} : \\
&+ : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} : \quad [A_2 I_1] \\
&+ : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} : \quad [A_2 II_1] \\
&+ : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} : \quad [B_2 I_1] \\
&+ : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} : \quad [B_2 II_1] \\
&+ : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} : \Big] \, |1_k^1 > \quad [B_2 III_1]
\end{aligned}
\qquad ([47])
$$

For the first term (without Wick contractions), the factorised part

$$< 1_p^2 \tilde{1}_{p'}^2 | : \hat{\phi}_1(x_1^\mu) \hat{\phi}_1(x_2^\mu) \hat{\phi}_1(x_3^\mu) : |1_k^1 >$$

can only contain the following vanishing terms,

$$
\begin{cases}
< 0^1 | \hat{\phi}_{1+} \hat{\phi}_{1+} \hat{\phi}_{1+} | 1_k^1 >= 0\,, \\
< 0^1 | \hat{\phi}_{1+} \hat{\phi}_{1+} \hat{\phi}_{1-} | 1_k^1 >= 0\,, \\
< 0^1 | \hat{\phi}_{1+} \hat{\phi}_{1-} \hat{\phi}_{1-} | 1_k^1 >= 0\,, \\
< 0^1 | \hat{\phi}_{1-} \hat{\phi}_{1-} \hat{\phi}_{1-} | 1_k^1 >= 0\,.
\end{cases}
$$

Regarding the second term (second line) above, we observe that [first renaming $x_2 \leftrightarrow x_3$]

$$\int\!\!\int\!\!\int d^4x_1\, d^4x_2\, d^4x_3 : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} :$$

$$= \int\!\!\int\!\!\int d^4x_1\, d^4x_3\, d^4x_2 : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2} :$$

$$= \int\!\!\int\!\!\int d^4x_1\, d^4x_2\, d^4x_3 : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} :$$

so that a factor 3 can take into account the basic position permutations. Similarly [first renaming $x_1 \leftrightarrow x_2$]

$$\int\!\!\int\!\!\int d^4x_1\, d^4x_2\, d^4x_3 : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\, \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} :$$

$$= \int\int\int d^4x_2\, d^4x_1\, d^4x_3 : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\ \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\ \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} :$$

$$= \int\int\int d^4x_1\, d^4x_2\, d^4x_3 : \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_2}\ \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_1}\ \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1|_{x_3} :$$

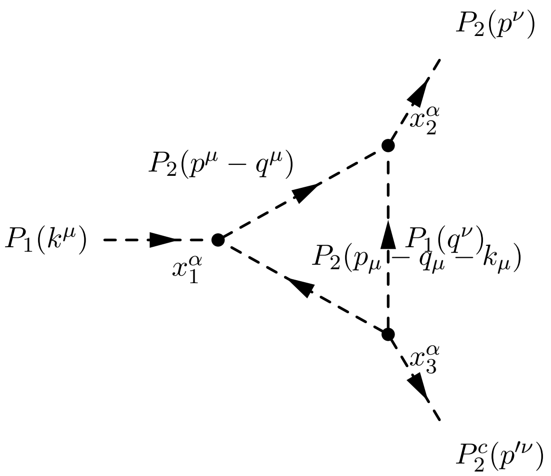
which could be shown directly as well (using time-ordering definition) and which leads here to a factor 2 for taking into account the order permutations. In conclusion, including the order permutations and position permutations is equivalent to multiply by a factor 2×3 which compensates exactly the factor $1/3!$ from the perturbation series form [as already considered in Eq.([47])]. Identical factors hold for the other terms, allowing to get rid of the same factor $1/3!$.

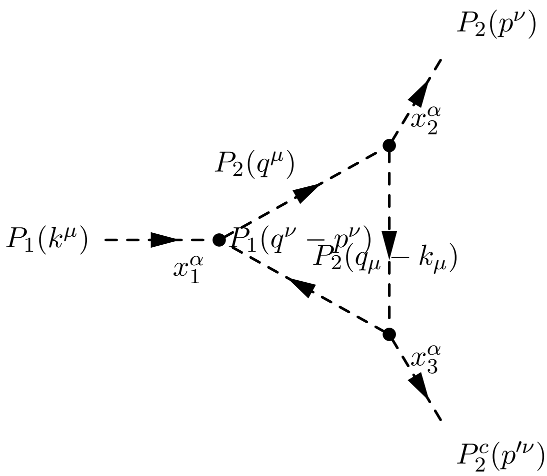
Now let us focus on the calculation of the last term (last line), namely the term B_2III_1, in Eq.([47]) [using in particular Eq.(37) to make the propagators arise]:

$$A^{(3)}_{B_2III_1} = (-ig)^3 < 1^2_p \tilde{1}^2_{p'}| \int d^4x_1\, d^4x_2\, d^4x_3\ (:)\hat{\phi}_2^\dagger|_+(x_2^\mu)\, \hat{\phi}_{2+}(x_3^\mu)\, \hat{\phi}_{1-}(x_1^\mu)(:)$$
$$\overbrace{\hat{\phi}_2^\dagger(x_1^\mu)\hat{\phi}_2(x_2^\mu)}^{iG_2(x_2^\mu - x_1^\mu)\mathbb{1}}\ \overbrace{\hat{\phi}_2(x_1^\mu)\hat{\phi}_2^\dagger(x_3^\mu)}^{iG_2(x_1^\mu - x_3^\mu)\mathbb{1}}\ \overbrace{\hat{\phi}_1(x_2^\mu)\hat{\phi}_1(x_3^\mu)}^{iG_1(x_2^\mu - x_3^\mu)\mathbb{1}}\ |1^1_k > . \qquad ([48])$$

<u>Feynman diagram for the term B_2III_1:</u>

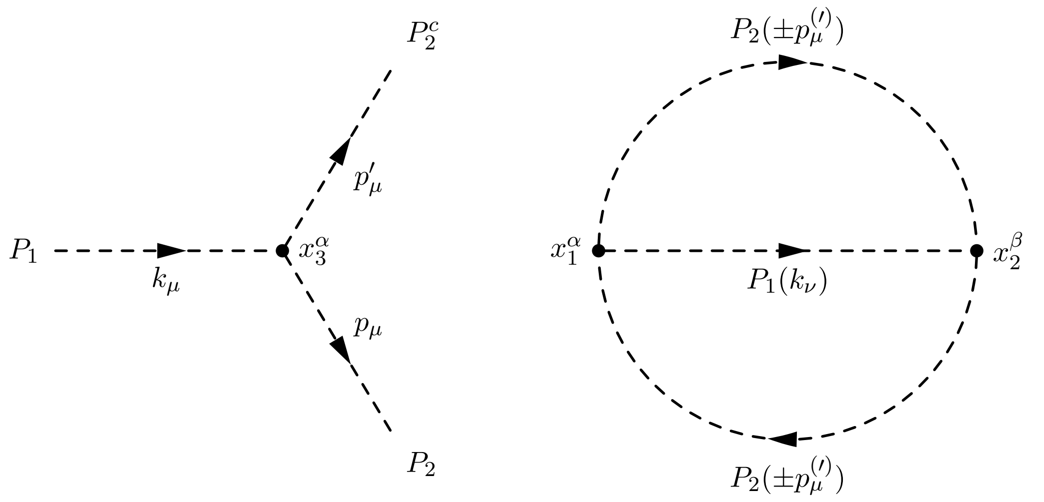
Assuming the 4-momentum conservation (to be demonstrated later on, as a result of the formalism), we can draw for instance the two following diagrams.



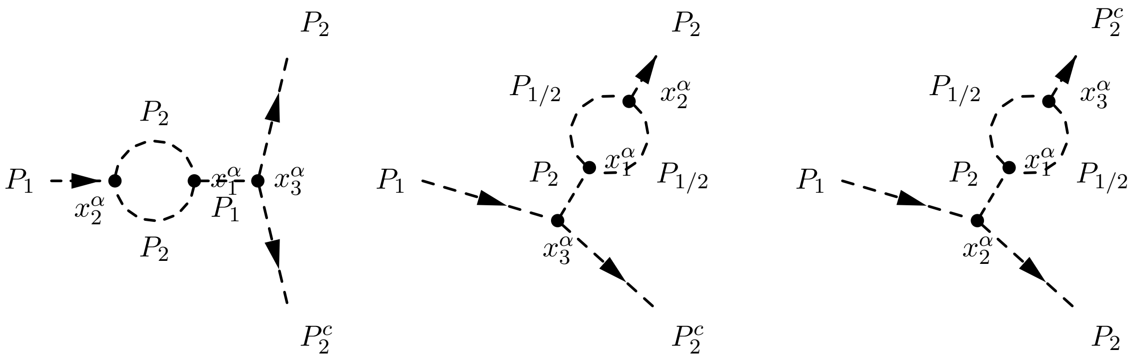


The two above diagrams are physically equivalent [same intermediate 4-momentum ranges span due to the summation over the new 4-momentum q^μ to be found in Eq.([52])].

<u>Feynman diagram for the term A_2I_1:</u>



<u>Feynman diagrams respectively for the terms A_2II_1, B_2I_1, B_2II_1:</u>



7.3 Amplitude calculation

Using Eq.(19) and Eq.(29), as well as the 3-momentum replacement (induced by the Dirac delta function integration) in the classical expression of the energy for the free field (fixed below Eq.([16]) for the real field):

$$q^0 = E_q \hat{=} \sqrt{\vec{q}^2 + m_1^2} \mapsto \sqrt{\vec{k}^2 + m_1^2} \hat{=} E_k = k^0 \,, \tag{39}$$

where the last equality (that will allow to recover a Lorentz product form k_x_1) is induced by the free and in turn classical nature of the initial particle [as stated above Eq.([33]) when we were describing the physical context of the decay reaction amplitude calculation], we find that [easier to first consider the 3 calculations in the separate Hilbert spaces, without compact notation with respect to the spaces associated to different (anti)particle species],

$$\begin{cases} < 0^1|\hat{\phi}_{1-}(x_1^\mu)|1_k^1 >=< 0^1| \int \frac{d^3q}{\sqrt{(2\pi)^3 2E_q}}\ \hat{a}_{1q}\, e^{-iq\_x} \sqrt{\frac{(2\pi)^3}{V}}\ \hat{a}^\dagger_{1k}\, |0^1 > \\ \qquad =< 0^1| \int \frac{d^3q}{\sqrt{2E_q}} \frac{e^{-iq\_x_1}}{\sqrt{V}} \left(\delta^{(3)}(\vec{q}-\vec{k}) + \hat{a}^\dagger_{1k}\hat{a}_{1q}\right)|0^1 > =< 0^1| \frac{1}{\sqrt{2E_k V}} e^{-ik\_x_1}|0^1 > \\ \hat{\phi}_{2-}(x_2^\mu)|1_p^2 > = \frac{1}{\sqrt{2E_pV}}\ e^{-ip\_x_2}|0^2 > \ \Rightarrow\ < 1_p^2|\hat{\phi}^\dagger_2|_+(x_2^\mu)|0^2 >= \frac{1}{\sqrt{2E_pV}}\ e^{ip\_x_2} < 0^2|0^2 > \\ \hat{\phi}^\dagger_2|_-(x_3^\mu)|\tilde{1}^2_{p'} > = \frac{1}{\sqrt{2E_{p'}V}}\ e^{-ip'\_x_3}|\tilde{0}^2 > \ \Rightarrow\ < \tilde{1}^2_{p'}|\hat{\phi}_{2+}(x_3^\mu)|\tilde{0}^2 >= \frac{1}{\sqrt{2E_{p'}V}}\ e^{ip'\_x_3} < \tilde{0}^2|\tilde{0}^2 > \end{cases} \qquad ([50])$$

which allows to write down the amplitude at first order, using the relation [3] $F.T.(1_x) = 2\pi\delta(p)$:

$$\text{Eq.([46])}:\ A^{(1)} = (-ig) \int d^4x\ \underbrace{< 0^2| < 0^{2c}|0^1 >}_{=1}\ \frac{e^{-ik\_x}}{\sqrt{2E_kV}} \frac{e^{ip\_x}}{\sqrt{2E_pV}} \frac{e^{ip'\_x}}{\sqrt{2E_{p'}V}}$$

$$= \boxed{(-ig)\ (2\pi)^4\ \delta^{(4)}(k_\nu - p_\nu - p'_\nu)\ \frac{1}{\sqrt{2E_kV}} \frac{1}{\sqrt{2E_pV}} \frac{1}{\sqrt{2E_{p'}V}}} \qquad ([51])$$

Therefore, we find the global **4-momentum conservation** mentioned previously [below Eq.([48])]. The dimensional analysis of the worked out amplitude goes like,

$$\left[A^{(1)}\right] = [g][E]^{-4} \left(\frac{1}{\sqrt{[E][E]^{-3}}}\right)^3 = [E][E]^{-4}[E]^3 = [E]^0 = 1\,.$$

Eq.([50]) also permits to write the term considered at the next non-vanishing higher order,

$$\text{Eq.([48])}:\ A^{(3)}_{B_2III_1} = \ (-ig)^3 \ \int \frac{d^4x_1\ d^4x_2\ d^4x_3}{\sqrt{V^3\, 2E_k\, 2E_p\, 2E_{p'}}}\ e^{-ik\_x_1+ip\_x_2+ip'\_x_3} \\ \times\ iG_2(-x_1+x_2)\ iG_2(x_1-x_3)\ iG_1(x_2-x_3)\ .$$

[3] The Fourier transformation ($F.T.$) involved in this formula undergoes the mathematical convention where neither π nor numerical factors are included in the transformation.

Using Eq.([29]bis), one can then rewrite this term making the $x_{1,2,3}$-dependences explicit:

$$\begin{aligned} A^{(3)}_{B_2III_1} &= (-ig)^3 \int \frac{d^4x_1\ d^4x_2\ d^4x_3}{\sqrt{V^3\,2E_k\,2E_p\,2E_{p'}}}\ e^{-ik.x_1+ip.x_2+ip'.x_3} \\ &\times \int \frac{d^4q_1\ d^4q_2\ d^4q_3}{(2\pi)^4\ (2\pi)^4\ (2\pi)^4}\ e^{-iq_1.(x_2-x_1)-iq_2.(x_1-x_3)-iq_3.(x_2-x_3)}\ iG_2(q_1^\nu)\ iG_2(q_2^\nu)\ iG_1(q_3^\nu) \end{aligned}$$

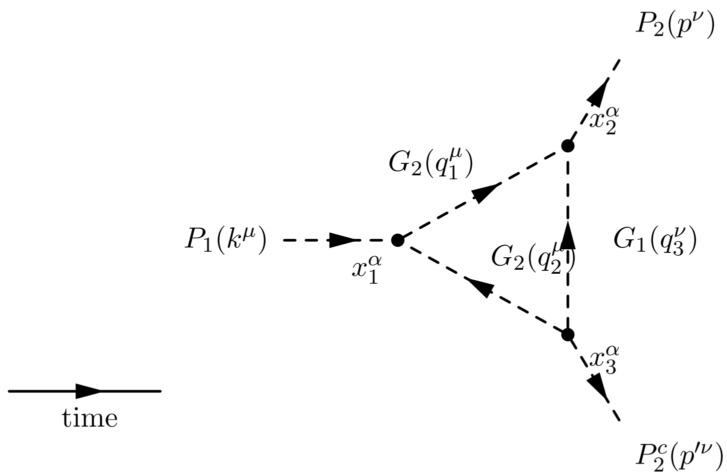
$$= (-ig)^3 \int \frac{d^4q_1\ d^4q_2\ d^4q_3}{\sqrt{V^3\,2E_k\,2E_p\,2E_{p'}}}\ \delta^{(4)}(k^\nu - q_1^\nu + q_2^\nu)\ \delta^4(-p^\nu + q_1^\nu + q_3^\nu)\ \underbrace{\delta^{(4)}(-p'^\nu - q_2^\nu - q_3^\nu)}_{\delta^{(4)}(k^\nu - p^\nu - p'^\nu)}$$
$$\times\ iG_2(q_1^\mu)\ iG_2(q_2^\mu)\ iG_1(q_3^\mu)$$

where we have used once again the equality, $F.T.(1_x) = 2\pi\delta(p)$, and also the specific choice of summing the three following relations,

$$\begin{cases} 0 = & k^\nu - q_1^\nu + q_2^\nu \\ 0 = & -p^\nu + q_1^\nu + q_3^\nu \\ 0 = & -p'^\nu - q_2^\nu - q_3^\nu \end{cases}$$

At this level, we find [4] the **4-momentum conservation** at each vertex mentioned previously [below Eq.([48])], as illustrated by the following Feynman diagram.

Feynman diagram:



$$A^{(3)}_{B_2III_1} = (-ig)^3 \int \frac{d^4q_1\ d^4q_3}{\sqrt{V^3\,2E_k\,2E_p\,2E_{p'}}}\ \delta^{(4)}(-p^\nu + q_1^\nu + q_3^\nu)\ \delta^4(k^\nu - p^\nu - p'^\nu)$$

[4] Up to the arrows.

$$\times\, iG_2(q_1^\mu)\; iG_2(q_1^\mu - k^\mu)\; iG_1(q_3^\mu)$$

$$= \boxed{\frac{(2\pi)^4\ (-ig)^3}{\sqrt{V^3\, 2E_k\, 2E_p\, 2E_{p'}}} \int \frac{d^4q}{(2\pi)^4}\, \delta^{(4)}(k^\mu - p^\mu - p'^\mu)\; iG_2(p^\mu - q^\mu)\; iG_2([p^\mu - q^\mu] - k^\mu)\; iG_1(q^\mu)} \quad ([52])$$

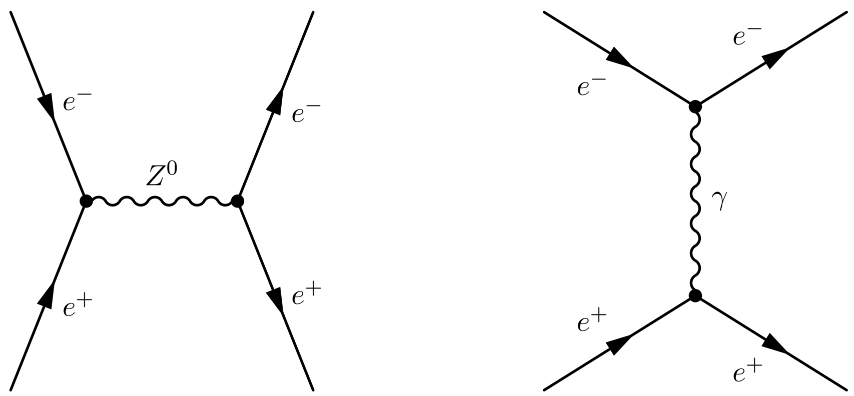
after renaming q_3^μ into q^μ in the last line. Here the continuous summation over the intermediate 4-momentum q^μ can be interpreted as a sum of probability amplitudes, all contributing to the studied amplitude of transition between initial and final quantum states. The dimensional analysis of Eq.([52]) goes like,

$$\left[A^{(3)}_{B_2III_1}\right] = [g]^3 \left(\frac{1}{\sqrt{[E][E]^{-3}}}\right)^3 [E]^4[E]^{-4}\left([E]^{-2}\right)^3 = [E]^3[E]^3[E]^4[E]^{-4}[E]^{-6} = [E]^0 = 1\,.$$

The intermediate 4-momenta $q_1^\mu, q_2^\mu, q_3^\mu$ (or q^μ) involved in the last Feynman diagram arise in the calculation from Fourier transformations and constitute thus mathematical variables which have no reason to be physically limited to classical relations, so that one can possibly have quantum fluctuations:

$$q^\mu q_\mu \neq M_1^2\,,\quad q_2^2 \neq m_2^2\,,\quad q_1^2 \neq m_2^2\,.$$

This result of the QFT formalism shows that the so-called '**loop**' ($\equiv$ closed intermediate momentum flow) appearing in this Feynman diagram [triangular loop here] represents a **quantum correction** to the amplitude (between the given initial and final states). This feature may be explicit diagrammatically. The precise amount of quantum fluctuation is driven by the Heisenberg principle of uncertainty. Intermediate 4-momenta also undergo quantum fluctuations in **tree-level** diagrams, as for instance the 4-momentum of the virtual Z-boson [or possibly heavy modes like new physics states of the kind Z'] exchanged in the s-channel of the electron-positron scattering ($e^+e^- \to e^+e^-$) experimentally studied at the LEP collider: pole resonance of the cross section corresponding to the classical situation, $p_Z^2 = M_Z^2$. Here are Feynman diagrams for this Bhabha scattering:



In contrast, the initial 4-momentum k^μ and final 4-momenta p^μ, p'^μ of the calculation come from the free field expressions [see Eq.(39)] (exponential dependences in Eq.([50]) changed to Dirac delta functions in the results of Eq.([51]),([52]) through Fourier transformation applications) and obey the classical relations,

$$k^\alpha k_\alpha = (k^0)^2 - (k^1)^2 - (k^2)^2 - (k^3)^2 = M_1^2\,, \quad p^{\underline{2}} = m_2^2\,, \quad p'^{\underline{2}} = m_2^2\,, \tag{40}$$

as stated above Eq.([33]) when describing the physical reaction context. Particles satisfying the classical relation of type (40) are called 'on-shell', 'real' or 'classical' (in contrast with 'off-shell', 'virtual' or 'quantum' particles) since this kind of relation describes geometrically an hyper-sphere in space-time [let us recall in comparison some generic R-radius sphere equation: $x^2 + y^2 + z^2 = R^2$].

Therefore, the conservation of the intermediate 4-momenta can be considered as a QFT result, in the sense that those 4-momenta do not constitute classical eigenvalues (as seen just above) of the type: $q^2_{\rm classical} =< \hat{Q}^{\underline{2}} >$, or $< \hat{H} >$ which is conserved in Quantum Mechanics as a consequence of the Ehrenfest theorem:

$$\frac{d}{dt}\langle\psi(t)|\hat{H}|\psi(t)\rangle = \frac{1}{i\hbar}\langle[\hat{H},\hat{H}]\rangle_{|\psi(t)\rangle} + \langle\frac{\partial\hat{H}}{\partial t}\rangle_{|\psi(t)\rangle} = 0 + 0 = 0\,,$$

remaining true within the Heisenberg picture [time-dependent operators].

7.4 Generalisation to Feynman rules

The obtained results generalise to Feynman rules which allow to understand and start easily the calculation of any amplitude at order (n), for transitions $|i> \to |f>$: it generically reads as,

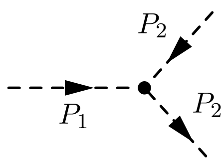
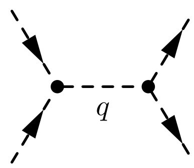
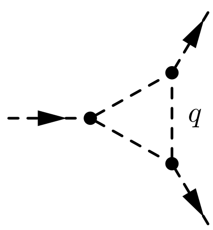
$$A^{(n)}_{if} \;=\; (2\pi)^4\,\delta^{(4)}\Big(\sum_i k_i^\nu - \sum_f p_f^\nu\Big)\;\Pi_i\,\frac{1}{\sqrt{2E_iV}}\;\Pi_f\,\frac{1}{\sqrt{2E_fV}}\;\;i\mathcal{M}^{(n)}_{if}$$

where the last amplitude part can be directly obtained via the Feynman rules given in the following table. The Feynman rules are thus powerful in the sense that those avoid to apply the full Wick's theorem approach for calculating transition amplitudes.

Some Feynman diagrams, like the one representing the contribution of the A_2II_1 term and drawn at the end of Section 7.2.3 (first one of the three horizontal diagrams: on the extreme left-side), exhibit some pure divergences (not only a pole inside some possibly convergent integration, as discussed in the propagator section). This diagram is called 1-Particle Reducible (1PR) since a cut of the intermediate P_1 particle leg (between the events x_1^α and x_3^α) would lead to two separate diagrams. The 4-momentum that is propagating along this leg is the same as the initial particle 4-momentum: k^μ, as induced by the 4-momentum conservation. Hence, the associated amplitude contains the following propagator,

$$G_1(k_\mu) \; = \; \frac{1}{k^{\underline{2}} - M_1^2}$$

with $k^{\underline{2}} = M_1^2$ due to the classical nature of the initial state [discussed at the end of previous subsection: see Eq.(40)]. Such divergent contributions to the amplitude must be 'renormalised'. The renormalisation process is beyond the scope of the present QFT course.

Name (*interpretation*)	Diagram pattern	$i\mathcal{M}$ content
Vertex (*coupling*)		"$-i\ g$" if $\mathcal{L}_I = -g\ \hat{\phi}_2^\dagger \hat{\phi}_2 \hat{\phi}_1$ ("$-i4!\ \lambda$" if $\mathcal{L}_I = -\lambda\ \hat{\phi}^4$)
Intermediate leg (*propagator*)		$iG(q_\mu) \; = \; i/(q^{\underline{2}} - m^2)$
Loop (*exchange*)		$\int d^4q\ /\ (2\pi)^4$

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