PROBLEM OF QUANTUM FIELD THEORY

s and t-channels

I) Propagators: [1]

1. We consider a quantised complex scalar field [2] $\phi(x^\nu)$. Express $<0|\mathcal{T}\left[\phi(x^\nu)\phi^\dagger(x'^\nu)\right]|0>$ in terms of field product mean values and the $\Theta(X)$ Heaviside step function [3]. $\mathcal{T}$ indicates the time-ordering of fields and x^ν is the coordinate 4-vector of the covariant formalism.
2. By using the field expressions and the canonical commutation relation, express the matrix element $<0|\phi(x^\nu)\phi^\dagger(x'^\nu)|0>$ as a function of a unique integral (over a three-momentum $\vec{p}$).
3. Express $\phi(x^\nu)|0>$ in term of an anti-particle quantum state. Deduce the form of $<0|\phi^\dagger(x^\nu)$.
4. Deduce the form of the mean value $<0|\phi^\dagger(x'^\nu)\phi(x^\nu)|0>$ from the previous question.
5. By using Questions 1-2-4, express $<0|\mathcal{T}\left[\phi(x^\nu)\phi^\dagger(x'^\nu)\right]|0>$ in terms of the Heaviside step function and an integral over a single three-momentum $\vec{p}$. Relate the obtained result to the *Green* function $G(x^\nu - x'^\nu)$.
6. Give (without demonstration) the similar link between the same *Green* function and a real scalar field $\phi(x^\nu)$.

II) Amplitudes:

1. Let us consider the interaction term in the following Lagrangian density, completing the free Lagrangian for 3 distinct real scalar fields ϕ_1, ϕ_2, ϕ_3 associated to spinless particles of respective masses m_1, m_2, m_3,

$$\mathcal{L}_{\text{Int.}}(x) \;=\; -\lambda\, \phi_1(x)\, \phi_2(x)\, \phi_3(x)\ ,$$

where we have adopted the simplified notation $\phi_i(x) \equiv \phi_i(x^\mu)$, μ being a *Lorentz* index.

   (a) How is called the λ parameter? Derive its energy dimension within the natural unit system. Show that this Lagrangian part is Hermitian (for λ real).

[1] The parts I and II can be treated independently.
[2] Fields and annihilation/creation **operators** are noted without accent, for simplicity: $\phi^{(\dagger)} \Leftrightarrow \hat{\phi}^{(\dagger)}$, $a^{(\dagger)} \Leftrightarrow \hat{a}^{(\dagger)}$.
[3] Defined by $\Theta(X) = 1$ for $X > 0$; 0 for $X < 0$; $1/2$ for $X = 0$.

(b) Give the associated Hamiltonian density $\mathcal{H}_{\text{Int.}}$ in terms of the fields and justify its form.

2. Based on the interaction defined right above, we are going to calculate the probability amplitude for the scattering reaction, among the two particle species named P_1, P_2 and described by the fields $\phi_{1,2}(x)$,

$$P_1(k_\mu) + P_2(k'_\mu) \to P_1(p_\mu) + P_2(p'_\mu)$$

where the associated 4-momenta are indicated in brackets.

(a) Calculate the first term of the amplitude $A^{(0)}$, from the S-matrix expansion in the perturbation approach, and comment it. We denote the initial quantum state via the compact notation, $|i\!>=|1^{(1)}_k 1^{(2)}_{k'}\!>$.

(b) Same question for the second term $A^{(1)}$ of the perturbative series. For that purpose, apply the *Wick* theorem and then focus on the action of the ϕ_3 operator.

3. Calculate the third term $A^{(2)}$ of the amplitude expansion. When applying the *Wick* theorem, justify shortly that only one term gives rise to a non-vanishing contribution. Show that this contribution is of the form,

$$A^{(2)} = (-i\lambda)^2 <\!f| \int d^4x_1 \int d^4x_2 :\!\phi_1(x_1)\phi_1(x_2)\!: \; :\!\phi_2(x_1)\phi_2(x_2)\!: \underbrace{\phi_3(x_1)\phi_3}(x_2) \; |i\!>.$$

4. The free scalar fields ($j = 1, 2, 3$) can be decomposed accordingly to,

$$\phi_j(x^\mu) = \int_{-\infty}^{+\infty} \frac{d^3q}{\sqrt{(2\pi)^3 2E_q(m_j)}} \left(a_{jq} e^{-iq.x} + a^\dagger_{jq} e^{iq.x} \right) = \phi_{j-}(x^\mu) + \phi_{j+}(x^\mu)$$

which involves the creation operator $a_{jq} \hat{=} a_j(\vec{q})$, the relativistic energy $E_q(m_j)$ and the *Lorentz* product within the covariant formalism $q.x = q_\mu x^\mu$.

(a) Express the normal-ordered product $:\!\phi_1(x_1)\phi_1(x_2)\!:$ exclusively in terms of $\phi_{1\pm}(x_{1,2})$.

(b) Using the lecture result, $\phi_{1-}(x)|1^{(1)}_q\!>= \frac{e^{-iq.x}}{\sqrt{2E_qV}}|0^{(1)}\!>$, argue that $\phi_{1-}(x_1)\phi_{1-}(x_2)|1^{(1)}_k\!>$ $= 0$. Deduce $<\!1^{(1)}_k|\phi_{1+}(x_1)\phi_{1+}(x_2)$.

5. Using Questions 3-4a-4b and the *Green* function definition, $\underbrace{\phi_3(x_1)\phi_3}(x_2) = iG_3(x_1 - x_2)\mathbb{1}$, calculate and develop the $A^{(2)}$ amplitude. Interpret each term obtained through a *Feynman* diagram. Indicate all the relevant information on those diagrams. Do you know the terminology for the configurations of such diagrams?

6. Using the lecture result recalled in Question 4b, simplify the following quantity,

$$\begin{aligned}\mathcal{Q} \;=\; (-i\lambda)^2 \;\; &<\!1^{(1)}_p 1^{(2)}_{p'}| \int d^4x_1 \int d^4x_2 \; i\,G_3(x_1 - x_2) \\ &\phi_{1+}(x_2)\phi_{1-}(x_1)\phi_{2+}(x_2)\phi_{2-}(x_1)\; |1^{(1)}_k 1^{(2)}_{k'}\!>\end{aligned}$$

7. Recall that the (4-momentum dependent) propagator $G_3(q^\mu)$ is related to $G_3(x^\mu - x'^\mu)$ through the *Fourier* transformations,

$$G_3(x^\mu - x'^\mu) \;=\; \int_{-\infty}^{+\infty} \frac{d^4q}{(2\pi)^4}\, e^{-iq.(x-x')}\, G_3(q^\mu)\ . \tag{1}$$

   (a) Rewrite the $\mathcal{Q}$ expression obtained in previous question by injecting the expression of Equation (1). Then integrate the resulting $\mathcal{Q}$ expression over the x_1^μ and x_2^μ coordinates.

   (b) Finally integrate the obtained $\mathcal{Q}$ expression over the 4-momentum (q^μ). Is the result compatible with what the *Feynman* rule method would have given? Make a physical comment about the *Dirac* peaks obtained.

8. Express $G_3(q^\mu)$ in terms of q^μ. Is it a divergent object? (in the present context)

9. Comment about the *Lorentz* square for each 4-momentum k^μ, k'^μ, p^μ and p'^μ. Justify your comment shortly. Provide the terminology for the corresponding particles.

***