PROBLEM OF QUANTUM FIELD THEORY

A scattering amplitude

1. **Preliminary calculations.**

   (a) The *Green* function – allowing to solve the *Klein-Gordon* equation with a source term for a field of mass m – satisfies the relation, $(\Box_x + m^2)G(x^\mu - x^{\mu\prime}) = -\delta^4(x^\mu - x^{\mu\prime})$, where $x^\mu \equiv (t, \vec{x})$ denotes the coordinate 4-vector [μ being a *Lorentz* index: $\mu = 0, 1, 2, 3$]. Use this relation to find out the expression of the 4-momentum function $G(q^\mu)$, defined through the *Fourier* transformations

$$G(x^\mu - x^{\mu\prime}) \;=\; \int_{-\infty}^{+\infty} \frac{d^4q}{(2\pi)^4}\, e^{-iq_{\cdot}(x-x')}\, G(q^\mu) \tag{1}$$

   using the compact notation, $q_{\cdot}x = q_\mu x^\mu$, for the *Lorentz* scalar product. How is called $G(q^\mu)$?

   (b) The free real (complex) scalar fields for a spinless particle of mass m_1 (m_2) can be decomposed accordingly to [1],

$$\phi_1(x^\mu) = \int_{-\infty}^{+\infty} \frac{d^3p}{\sqrt{(2\pi)^3 2E_p(m_1)}} \left(a_{1p}e^{-ip_{\cdot}x} + a^\dagger_{1p}e^{ip_{\cdot}x}\right) = \phi_{1-}(x^\mu, a_1) + \phi_{1+}(x^\mu, a^\dagger_1)$$
$$\phi_2(x^\mu) = \int_{-\infty}^{+\infty} \frac{d^3p}{\sqrt{(2\pi)^3 2E_p(m_2)}} \left(a_{2p}e^{-ip_{\cdot}x} + \tilde{a}^\dagger_{2p}e^{ip_{\cdot}x}\right) = \phi_{2-}(x^\mu, a_2) + \phi_{2+}(x^\mu, \tilde{a}^\dagger_2)$$
$$\phi^\dagger_2(x^\mu) = \int_{-\infty}^{+\infty} \frac{d^3p}{\sqrt{(2\pi)^3 2E_p(m_2)}} \left(\tilde{a}_{2p}e^{-ip_{\cdot}x} + a^\dagger_{2p}e^{ip_{\cdot}x}\right) = \phi^\dagger_{2-}(x^\mu, \tilde{a}_2) + \phi^\dagger_{2+}(x^\mu, a^\dagger_2)$$

   where $a_p \hat{=} a(\vec{p})$ and $E_p(m)$ denotes the relativistic energy. What differs in the physical nature of a_2 and $\tilde{a}_2$? Express the normal-ordered product $:\phi_1\phi_1:$ exclusively in terms of ϕ_{1-} and ϕ_{1+}.

   (c) A 1-particle state of the species n'2 associated to the above field ϕ_2 is defined per reference volume V as, $|1^{(2)}_k\rangle = \sqrt{(2\pi)^3/V}\, a^\dagger_{2k}|0\rangle$. Show that, $\phi_{2-}(x^\mu, a_2)|1^{(2)}_k\rangle = e^{-ik_{\cdot}x}/\sqrt{2E_kV}\,|0\rangle$, with the help of the canonical commutator between the annihilation and creation operators.

[1] The field and annihilation/creation **operators** are noted without accent, for simplicity: $\phi^{(\dagger)} \Leftrightarrow \hat{\phi}^{(\dagger)}$, $a^{(\dagger)} \Leftrightarrow \hat{a}^{(\dagger)}$.

(d) The previous result can be extended to two 4-momenta $k^\mu, k^{\mu\prime}$: find the second term in this equation

$$\phi_{2-}(x,a_2)\,\phi_{2-}(x',a_2)\,|1_k^{(2)}>\otimes|1_{k'}^{(2)}> = \frac{e^{-ik.x}}{\sqrt{2E_kV}}\frac{e^{-ik'.x'}}{\sqrt{2E_{k'}V}}\,|0> + (2nd\,\text{term})\;, \quad (2)$$

where the *Lorentz* indices are omitted to simplify the notations. Verify that the Hermitian conjugate of this equality gives rise to (choosing other 4-momentum names)

$$<1_p^{(2)}|\otimes<1_{p'}^{(2)}|\;\phi_{2+}^\dagger(x,a_2^\dagger)\,\phi_{2+}^\dagger(x',a_2^\dagger) \;=<0|\,\frac{e^{ip.x}}{\sqrt{2E_pV}}\frac{e^{ip'.x'}}{\sqrt{2E_{p'}V}} + (2nd\,\text{term})^\dagger. \quad (3)$$

2. Let us now consider the interaction term in the following Lagrangian density, completing the free Lagrangian [2] for the fields ϕ_1, ϕ_2 introduced in Question n'1b,

$$\mathcal{L}_{\text{Int.}}(x) \;=\; -\lambda\,\phi_2^\dagger(x)\,\phi_2(x)\,\phi_1(x)\;.$$

(a) How is called the λ parameter ? Justify its dimension. Show that this Lagrangian is Hermitian (for λ real).

(b) Derive the associated Hamiltonian $\mathcal{H}_{\text{Int.}}$ in terms of the fields.

3. Based on the interaction defined right above, we are going to calculate the probability amplitude for the scattering reaction, among particles named P_2 and described by the field $\phi_2(x)$,

$$P_2(k_\mu)+P_2(k'_\mu)\to P_2(p_\mu)+P_2(p'_\mu)$$

where the associated 4-momenta are indicated in brackets. In perturbation theory, the first non-vanishing contribution to this amplitude reads as ($\mathcal{T}$ indicates the time-ordering),

$$A^{(2)} \;=\; \tfrac{(-i\lambda)^2}{2!}\;<1_p^{(2)}|<1_{p'}^{(2)}|\int d^4x_1\int d^4x_2 \quad (4)$$
$$\mathcal{T}\Big[:\phi_2^\dagger(x_1)\phi_2(x_1)\phi_1(x_1):\;:\phi_2^\dagger(x_2)\phi_2(x_2)\phi_1(x_2):\Big]\;|1_k^{(2)}>|1_{k'}^{(2)}>\;.$$

(a) Show that the first term $A^{(0)}$, from the S-matrix expansion, vanishes. One can use considerations on the quantum state normalisations (assuming that all four 4-momenta are different from each other).

(b) Show that the second term $A^{(1)}$ of the perturbative series is also vanishing. For that purpose, focus on the action of the ϕ_1 operator on the vacuum state.

4. The matrix element of Equation (4) is equal to

$$A^{(2)} \;=\; \tfrac{(-i\lambda)^2}{2!}\;<1_p^{(2)}|<1_{p'}^{(2)}|\int d^4x_1\int d^4x_2 \quad (5)$$
$$\mathcal{T}\Big[\phi_2^\dagger(x_1)\phi_2(x_1)\phi_1(x_1)\;\;\phi_2^\dagger(x_2)\phi_2(x_2)\phi_1(x_2)\Big]_{\text{e.}\cancel{\text{t}}\text{.c.}}\;|1_k^{(2)}>|1_{k'}^{(2)}>$$

where the crossed out subscript "e.t.c." stands for "*no* equal time (*Wick*) contraction". Explain why the term without any *Wick* contraction, which arises when applying the *Wick*'s theorem to calculate the time-ordered product in Equation (5), does not contribute to $A^{(2)}$. Make use of the end of Question n'1b.

[2] With, $m_1 \neq m_2$.

5. Comment about the contractions $\underbracket{\phi_2\phi_2}$, $\underbracket{\phi_2^\dagger\phi_2^\dagger}$ and $\underbracket{\phi_1\phi_2^{(\dagger)}}$ possibly coming out from Equation (5).

6. Only the following term with a unique pair of contracted fields, issued from the *Wick*'s theorem application,

$$\begin{aligned} A^{(2)} \;=\; & \tfrac{(-i\lambda)^2}{2!} \;\; < 1_p^{(2)}| < 1_{p'}^{(2)}| \int d^4x_1 \int d^4x_2 \qquad (6) \\ & \left(: \phi_2^\dagger(x_1)\phi_2(x_1)\underbracket{\phi_1(x_1)\;\; \phi_2^\dagger(x_2)\phi_2(x_2)\phi_1}(x_2) : + \,\text{perm.} \right) |1_k^{(2)} > |1_{k'}^{(2)} > , \end{aligned}$$

contributes effectively to the amplitude $A^{(2)}$. Provide a short explanation of this feature based on the effects of the fields $\phi_2^{(\dagger)}$ on the states.

7. How many permutations of the *Wick* contraction can occur in Equation (6) ?

8. Justify that Equation (6) can be recast into (omitting to write explicitly the $a_2^{(\dagger)}$ dependences),

$$\begin{aligned} A^{(2)} \;=\; & (-i\lambda)^2 \;\; < 1_p^{(2)}| < 1_{p'}^{(2)}| \int d^4x_1 \int d^4x_2 \qquad (7) \\ & iG(x_1 - x_2)\; \phi_{2+}^\dagger(x_1)\phi_{2+}^\dagger(x_2)\phi_{2-}(x_1)\phi_{2-}(x_2)\; |1_k^{(2)} > |1_{k'}^{(2)} > \;. \end{aligned}$$

9. Interpret physically Equation (7) through a *Feynman* diagram illustrating [only] the initial/final/intermediate particle legs (denoted as $P_{1,2}$), momentum flow (with arrows on the legs) and vertex positions. Is this diagram *1-particle reducible* ? Could you draw the equivalent diagram with anti-particles only ($\tilde{P}_2$) ?

10. Draw the *Feynman* diagrams for the 1-loop quantum corrections to the studied reaction. Indicate exclusively the propagating particles ($P_{1,2}$).

11. Rewrite Equation (7) by using Equation (1) and Equations (2)-(3) [restricting oneself to the explicit first terms there]. Respect their variable orderings.

12. Integrate over x_1^μ and x_2^μ, the amplitude obtained in the previous question, thanks to the *Dirac* peak definition,

$$\delta^4(K^\mu) \;=\; \int_{-\infty}^{+\infty} \frac{d^4x}{(2\pi)^4}\, e^{-iK.x} \;.$$

Use the informations, from the two *Dirac* peaks so obtained in $A^{(2)}$, to indicate the 4-momenta (k,k',p,p' and an intermediate momentum say q with its arrow of flow) on the diagram of Question n'9. Is the intermediate exchanged particle off-shell ? Why ?

13. Perform the last integration remaining in the $A^{(2)}$ expression found in previous question. Which physical principle is predicted from the obtained *Dirac* peak ? Are the initial/final particles on-shell ? (invoke a formal argument)

14. Can you now interpret diagrammatically the $2nd$ term of Equation (2) ?

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