PROBLEM OF PARTICLE PHYSICS

Dirac matrix representations, quantum velocity and charge conjugation

(the 4 problems can be treated independently)

1. The mean value of the Hamiltonian operator restricted to the Hilbert space of spin states is, $u^{(1)\dagger}\hat{H}_s u^{(1)}$, where the normalised Dirac spinor reads as, $u^{(1)} = N_1(1, 0, \frac{p_3}{E+m}, \frac{p_1+ip_2}{E+m})^t$. m, E and $\vec{p}$ are respectively the fermion mass, energy and momentum eigenvalues. The Hamiltonian is given [1] by, $\hat{H}_s = \hat{\vec{\alpha}}.\vec{p} + \hat{\beta}m$, where the $\hat{\alpha}$, $\hat{\beta}$ matrices are considered within the Dirac-Pauli representation.

   (a) Calculate explicitly $u^{(1)\dagger}\hat{H}_s u^{(1)}$ and give the result as a function of N_1, E, $\vec{p}^2$ and m. What is the E eigenvalue sign?

   (b) Provide the previous result in a simplified form, involving exclusively N_1, E and m, by making use of the relativistic expresion, $E^2 = \vec{p}^2 + m^2$. Then inject $|N_1|^2 = E + m$ in the final result.

   (c) Was it possible to guess this result directly by recalling that $u^{(1)}$ is an eigenstate of H_s and using the $u^{(n)}$ ortho-normalisation condition?

2. We consider generically the possible transformations of Dirac matrix representations.

   (a) Verify that the Dirac matrix anti-commutation relation, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}$, is invariant under the change of basis defined through the U matrix as $\gamma^{\mu\prime} = U\gamma^\mu U^{-1}$.

   (b) Which property must fulfill U so that the transformed matrix $\gamma^{0\prime}$ satisfies the relation $\gamma^{0\prime\dagger} = \gamma^{0\prime}$ (as γ^0 does)?

   (c) Check that the spin observable [2] commutation relation

   $$[\hat{S}_i, \hat{S}_j] = i\sum_{k=1}^{3} \epsilon_{ijk}\,\hat{S}_k$$

   is invariant under the same change of basis: $\hat{S}'_i = U\hat{S}_i U^{-1}$.

   (d) Show that the spin eigenvalues are unchanged after the change of basis which acts on the spin eigenstates as $U|\eta\rangle = |\eta'\rangle$. (for this purpose, consider general $\hat{S}'_i$ eigenvalues)

[1] Throughout this examination we use the natural unit system where $\hbar = c = 1$.
[2] Defined within the 4-dimensional Hilbert space including the anti-particle state.

3. Let us discuss a quantum velocity defined within the whole Hilbert space $\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_s$ of momentum and spin states.

   (a) Using the Ehrenfest theorem, demonstrate the time derivative expression
   $$\frac{d}{dt}\langle\psi|\hat{X}|\psi\rangle = -i\langle\psi|[\hat{X},\hat{H}]|\psi\rangle$$
   where $|\psi\rangle = |\psi(t)\rangle \in \mathcal{H}$, $\hat{X}$ is the position observable along the first physical axis and $\hat{H}$ the Hamiltonian (both defined in $\mathcal{H}$).

   (b) Express the commutator $[\hat{X},\hat{H}]$ in terms of $\hat{\vec{\alpha}}$ components only, thanks to the expression, $\hat{H} = \hat{\vec{\alpha}}.\hat{\vec{P}} + \hat{\beta}m$, where now [3] $\hat{\vec{P}}$ is the momentum operator of $\mathcal{H}_x$. Make use of the position and momentum commutator.

   (c) Deduce $\frac{d}{dt}\langle\psi|\hat{X}|\psi\rangle$ from the two previous questions.

   (d) Then what does become $\frac{d}{dt}\langle\psi|\hat{X}|\psi\rangle$ for a state $|\psi\rangle = |p\rangle \otimes |\alpha\rangle$ (with $|p\rangle$ a normalised momentum eigenstate and $|\alpha\rangle$ a normalised $\hat{\alpha}_1$ eigenstate in $\mathcal{H}_s$)? What are the $\hat{\alpha}_1$ eigenvalues? Deduce a physical interpretation of the result.

4. We consider the charge conjugate operator C.

   (a) Show from the fundamental C property that $\gamma^0 C \gamma^0 = -C$.

   (b) Express $\left(\bar{\psi}C\gamma^\mu\psi\right)^\dagger$, ψ being the full Dirac spinor, exclusively in terms of ψ, $\bar{\psi}$, C and γ^μ.

***

[3] In contrast with the first exercise.