PROBLEM OF PARTICLE PHYSICS

The CPT theorem

1. **Charge conjugation.-** Let C denote the charge conjugate operator.
   (a) Show from a known C property that $\gamma^k C = -C(\gamma^k)^t$ with spatial Lorentz indices among $k = 1, 2, 3$ for the Dirac matrices.
   (b) Similarly, demonstrate the anti-commutator relation $\{\gamma^0, C\} = 0$.

2. **Parity action.-** The space parity changes the sign of each spatial coordinate $x^{1,2,3}$.
   (a) Give the corresponding $\Lambda^{\mu}_{\cdot\,\nu}$ Lorentz matrix.
   (b) Verify that the parity operator acting on the spinorial Hilbert space, $P = \gamma^0$, satisfies well the covariant relation, $\Lambda^{\mu}_{\cdot\,\nu}\,\gamma^{\nu} = P^{-1}\gamma^{\mu}\,P$, for $\mu = k$ [$k = 1, 2$ or 3] only.
   (c) Is this P operator Hermitian? Calculate P^2 and comment physically the result.

3. **Time reflection.-** The Lorentz matrix for the reflection on the time coordinate obviously reads as,

$$\tilde{\Lambda}^{\mu}_{\cdot\,\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} .$$

   (a) Express [1] the time-reflected 4-vector x'^{α} in terms of this $\tilde{\Lambda}^{\mu}_{\cdot\,\nu}$ matrix and the initial x^{α}. Deduce the new coordinates, t', x', y', z', as functions of the initial ones: t, x, y, z.
   (b) Using the covariant relation, $\tilde{\Lambda}^{\mu}_{\cdot\,\nu}\,\gamma^{\nu} = T^{-1}\gamma^{\mu}\,T$, where T is the time reflection operator acting on the spinorial Hilbert space, express the quantities $T^{-1}\gamma^0 T$ and $T^{-1}\gamma^k T$ ($k = 1, 2, 3$) in terms of Dirac matrices.
   (c) We consider the following operator T,

$$T = \begin{pmatrix} 0 & -i\,\mathbb{1}_{2\times 2} \\ i\,\mathbb{1}_{2\times 2} & 0 \end{pmatrix} . \tag{1}$$

   Is this T operator Hermitian? Calculate T^2 and comment physically the result.
   (d) Based on Equation (1), calculate [2] the anti-commutators $\{T, \beta\}$ and $\{T, \alpha^k\}$ ($k = 1, 2, 3$) within the Dirac-Pauli representation for the matrices β and α^k.

[1] Choosing consistently each Lorentz index.
[2] β and α^k are related to the Dirac matrices by $\gamma^{\nu} = (\gamma^0, \gamma^k) = (\beta, \beta\alpha^k)$.

(e) Same question but now within the so-called Weyl representation.

(f) Deduce from previous question the anti-commutator $\{T, \gamma^0\}$ and commutator $[T, \gamma^k]$.

(g) Given the results of Question 3f, does the T operator suggested in Equation (1) respect the two conditions obtained in Question 3b? Conclude.

4. **CPT transformation.-** The spinor ψ represents a solution of the Dirac equation, with $\bar{\psi} \hat{=} \psi^\dagger \gamma^0$.

   (a) $\overline{(TP\psi_c)}(TP\psi_c)$ represents the term $\bar{\psi}\psi$ transformed under charge conjugation, parity action and time reflection (CPT) [3]. Give $\overline{(TP\psi_c)}(TP\psi_c)$ as a function exclusively of $C^{(\dagger)}$, γ^0, ψ^t and $\psi^\star$. Use the definition $\psi_c = C\gamma^0\psi^\star$ and preliminary results (from Questions 2 and 3).

   (b) Use the previous question, Question 1 and a $C^\dagger$ property in order to express $\overline{(TP\psi_c)}(TP\psi_c)$ in terms of $\bar{\psi}$ and ψ only. For this purpose, calculate first $\bar{\psi}^t$.

   (c) Similarly, express $\overline{(TP\psi_c)}\gamma^0(TP\psi_c)$ as a function exclusively of γ^0, $\bar{\psi}$ and ψ. Provide only the final result and the changes with respect to the calculation of Questions 4a-4b.

   (d) Same question for $\overline{(TP\psi_c)}\gamma^k(TP\psi_c)$ ($k = 1, 2, 3$) as a function exclusively of γ^k, $\bar{\psi}$, ψ.

   (e) In the same way [4], express $\overline{(TP\psi_c)}(-\partial_\mu)(TP\psi_c)$ in terms of $\bar{\psi}\partial_\mu\psi$. Once more, give only the result and the differences with respect to the calculation in 4a and 4b.

   (f) Deduce directly $\overline{(TP\psi_c)}\gamma^\mu(-\partial_\mu)(TP\psi_c)$ from the 3 previous questions.

   (g) Conclude about the CPT transformation effect on the considered Lorentz invariant terms of a possible Lagrangian density.

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[3] The wave function part $\langle x_\mu | p^\mu \rangle$ of the spinor ψ is invariant under both the parity action and time reflection.

[4] Noting the 4-vector derivative $\partial_\mu = \frac{\partial}{\partial x^\mu}$.