# Equations aux dérivées partielles

La bibliothèque PDEtools permet de résoudre des équations aux dérivées partielles, et de visualiser les résultats.

## I Transferts de chaleur dans une sphère: la cuisson d'une dinde.

On considère une sphère de rayon R et de diffusivité thermique a. L'équation de la chaleur dans la sphère s'écrit:

$$\left(\frac{\partial^2}{\partial r\,\partial r}\theta(r,t)\right)+\frac{2\left(\frac{\partial}{\partial r}\theta(r,t)\right)}{r}-\frac{1\left(\frac{\partial}{\partial t}\theta(r,t)\right)}{a}=0.$$

On fera le changement de variable θ(r,t) = u(r,t)/r. Trouver l'équation pour u.

```
> restart;
```

```
> pde1:=diff(theta(r,t),r,r)+(2/r)*diff(theta(r,t),r)-(1/a)*
  diff(theta(r,t),t)=0;
```

$$pde1 := \left(\frac{\partial^2}{\partial r^2}\theta(r,t)\right)+\frac{2\left(\frac{\partial}{\partial r}\theta(r,t)\right)}{r}-\frac{\frac{\partial}{\partial t}\theta(r,t)}{a}=0$$

```
> pde2:=subs(theta(r,t)=u(r,t)/r,pde1);
```

$$pde2 := \left(\frac{\partial^2}{\partial r^2}\left(\frac{u(r,t)}{r}\right)\right)+\frac{2\left(\frac{\partial}{\partial r}\left(\frac{u(r,t)}{r}\right)\right)}{r}-\frac{\frac{\partial}{\partial t}\left(\frac{u(r,t)}{r}\right)}{a}=0$$

```
> pde3:=expand(pde2);
```

$$pde3 := \frac{\frac{\partial^2}{\partial r^2}u(r,t)}{r}-\frac{\frac{\partial}{\partial t}u(r,t)}{a\,r}=0$$

```
> pde4:=simplify(pde3*r);
```

$$pde4 := \frac{\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)a-\left(\frac{\partial}{\partial t}u(r,t)\right)}{a}=0$$

```
>
```

On suppose que la sphère est initialement à une température uniforme $\theta_0$

A t=0, on la met dans un gaz à la température $\theta_1$. Le coefficient d'échange entre la sphère et le gaz est h=30 en SI. La conduction thermique dans la sphère vaut $\lambda$.

Ecrire la condition aux limites en r=R:

$-\lambda\left(\frac{\partial}{\partial r}\theta\right) = h\,(\theta(R,t) - \theta_1)$.

On pourra regarder la doc pour pdsolve,numeric.

On assimile la sphère à une dinde, que l'on chauffe dans un four.

Choisir des valeurs réalistes pour $\theta_0$ et $\theta_1$.

Pour a=5.10$^{\{-7\}}$m2/s, λ=0.5 W/m.K, R=15cm, résoudre l'équation de la chaleur avec pdsolve, et tracer le profil de température pour t=5,50, et 90 mn.

Combien de temps faut-il pour que le coeur de la dinde soit à 70 degrés Celsius?

## Résolution en utilisant la variable u.

```
> CL:={u(0,t)=0,lambda*D[1](u)(R,t)/R-u(R,t)/(R*R)+h*(u(R,t)
  /R-theta[1]),u(r,0)=theta[0]*r};
```

$$CL := \left\{u(r,0) = \theta_0\, r,\ u(0,t) = 0,\ \frac{\lambda\, D_1(u)(R,t)}{R} - \frac{u(R,t)}{R^2} + h\left(\frac{u(R,t)}{R} - \theta_1\right)\right\}$$

Rque: on a une condition supplémentaire sur u(0,t) pour assurer que la température ne diverge pas en r=0.

```
> CL2:=subs(lambda=0.5, h=30, R=0.15, theta[0]=20,
  theta[1]=180,CL);
```

$$CL2 := \{u(0,t) = 0,\ u(r,0) = 20\, r,\ 3.333333334\, D_1(u)(0.15,t) + 155.5555556\, u(0.15,t) - 5400\}$$

$$\lambda := 0.1$$

$$a := 1$$

$$h := 30$$

$$R := 0.15$$

$$\theta_0 := 20$$

$$\theta_1 := 200$$

```
> pde5:=subs(a=0.0000005,pde4);
```

$$pde5 := 1.000000000\left(\frac{\partial^2}{\partial r^2}u(r,t)\right) - 0.2000000000\ 10^7\left(\frac{\partial}{\partial t}u(r,t)\right) = 0$$

```
> pdsolve(pde5);
```

$$(u(r,t) = _F1(r)\,_F2(t))\ \&\text{where}$$

$$\left[\left\{\frac{d^2}{dr^2}_F1(r) = _c_1\,_F1(r),\ \frac{d}{dt}_F2(t) = 0.5000000000\ 10^{-6}\,_c_1\,_F2(t)\right\}\right]$$

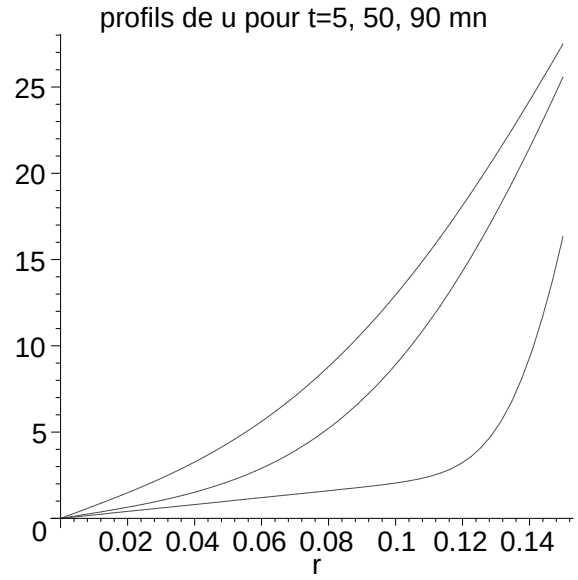
```
> sol1:=pdsolve(pde5,CL2,numeric,time=t,timestep=1);
```

*sol1* := **module**() **export** *plot*, *plot3d*, *animate*, *value*, *settings*; ... **end module**

```
> profil5:=sol1:-plot(t=300,numpoints=50);
```

*profil5* := PLOT(CURVES(Array(1..50, 1..2, [[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[
..],[...],[...],[...],[...],[...],[...],[...],[...],[...

```
      pe = float[8], storage = rectangular, order = C_order)                    ),
      COLOUR(RGB, 1.00000000, 0., 0.), AXESLABELS("r", ""))
> profil50:=sol1:-plot(t=3000,numpoints=50);
profil50 := PLOT(CURVES(Array(1..50, 1..2, [[...],[...],[...],[...],[...],[...],[...],[...]\[...],
      [...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[\..],[...]
      ..],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[..\]], data
      ype = float[8], storage = rectangular, order = C_order)                    ),
      COLOUR(RGB, 1.00000000, 0., 0.), AXESLABELS("r", ""))
> profil90:=sol1:-plot(t=5400,numpoints=50);
profil90 := PLOT(CURVES(Array(1..50, 1..2, [[...],[...],[...],[...],[...],[...],[...],[...]\[...],
      [...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[\..],[...]
      ..],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[...],[..\]], data
      ype = float[8], storage = rectangular, order = C_order)                    ),
      COLOUR(RGB, 1.00000000, 0., 0.), AXESLABELS("r", ""))
> plots[display]({profil5,profil50,profil90},title=`profils
  de u pour t=5, 50, 90 mn`);
```




```
> prof50:=sol1:-value(t=3000);
```

*prof50* := **proc**(*x_pde*) ... **end proc**

```
> prof50(.01);
```

[*r* = 0.01, *t* = 3000., u(*r*, *t*) = 0.309302766389220608]

```
> te50:=proc(x)::double;
> rhs(prof50(x)[3])/(x);
> end proc;
```

*te50* := **proc**(*x*)::*double*; rhs(prof50(*x*)[3]) / *x* **end proc**

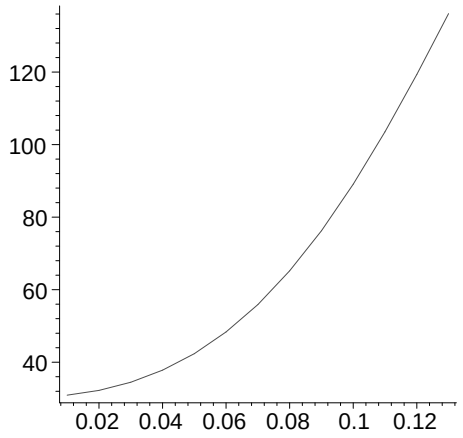
```
> te50 := proc (x)::double; rhs(prof50(x)[3])/x end proc;
```

*te50* := **proc**(*x*)::*double*; rhs(prof50(*x*)[3]) / *x* **end proc**

```
> profilT50:=[seq([t/100,te50(t/100)],t=1..13)];
```

$$profilT50 := \left[\left[\frac{1}{100}, 30.93027664\right], \left[\frac{1}{50}, 32.22570421\right], \left[\frac{3}{100}, 34.47513287\right], \left[\frac{1}{25}, 37.80045848\right], \left[\frac{1}{20}, 42.36567826\right], \left[\frac{3}{50}, 48.34543042\right], \left[\frac{7}{100}, 55.90676329\right], \left[\frac{2}{25}, 65.18450989\right], \left[\frac{9}{100}, 76.24677137\right], \left[\frac{1}{10}, 89.07292909\right], \left[\frac{11}{100}, 103.5217442\right], \left[\frac{3}{25}, 119.3281607\right], \left[\frac{13}{100}, 136.1086703\right]\right]$$

```
> plot(profilT50);
```




```
[ >
```

## Résolution en utilisant la variable theta

```
> CLT:={lambda*D[1](theta)(R,t)+h*(theta(R,t)-theta[1]),thet
  a(r,0)=theta[0],D[1](theta)(0,t)=0};
```

$$CLT := \{\theta(r, 0) = \theta_0, D_1(\theta)(0, t) = 0, \lambda D_1(\theta)(R, t) + h(\theta(R, t) - \theta_1)\}$$

```
> CLT2:=subs(lambda=0.5, h=30, R=0.15, theta[0]=20,
  theta[1]=180,CLT);
```

$$CLT2 := \{D_1(\theta)(0, t) = 0, 0.5 D_1(\theta)(0.15, t) + 30\theta(0.15, t) - 5400, \theta(r, 0) = 20\}$$

```
> pde6:=subs(a=0.0000005,pde1);
```

$$pde6 := \left(\frac{\partial^2}{\partial r^2}\theta(r, t)\right) + \frac{2\left(\frac{\partial}{\partial r}\theta(r, t)\right)}{r} - 0.2000000000\ 10^7\left(\frac{\partial}{\partial t}\theta(r, t)\right) = 0$$

```
> pdsolve(pde6);
```

$$(\theta(r, t) = _F1(r)\ _F2(t))\ \&\text{where}$$

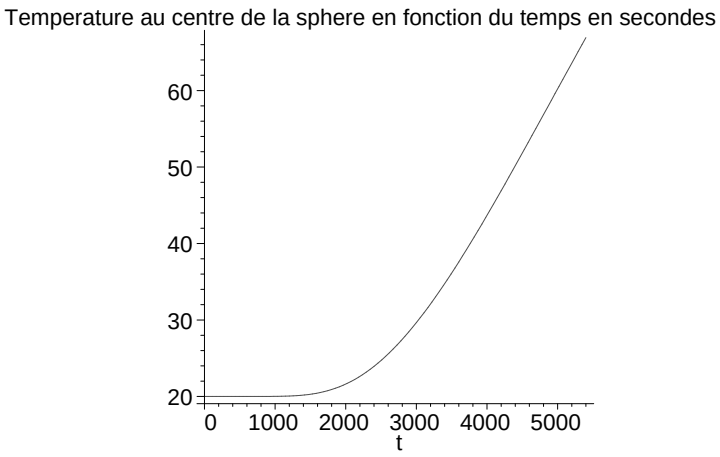
$$\left[\left\{\frac{d}{dt}_F2(t) = 0.5000000000\ 10^{-6}\ _c_1\ _F2(t), \frac{d^2}{dr^2}_F1(r) = _c_1\ _F1(r) - \frac{2\left(\frac{d}{dr}_F1(r)\right)}{r}\right\}\right]$$

```
> sol2:=pdsolve(pde6,CLT2,numeric,time=t,timestep=1);
```

$$sol2 := \textbf{module}()\ \textbf{export}\ plot, plot3d, animate, value, settings;\ \ldots\ \textbf{end module}$$

```
> profilT5:=sol2:-plot(t=300,numpoints=50);
```

*profilT5* := PLOT(CURVES(Array(1..50, 1..2, [[...],[...],[...],[...],[...],[...],[...],[...]],[...],
[...],[...],[...],[...],[...],[...],[...],




```
> Tcoeur2:=sol2:-value(r=0,t=0..5400);
```

$$Tcoeur2 := \mathbf{proc}(x_pde) \ \ldots \ \mathbf{end\ proc}$$

```
>
```