Study of $\eta \rightarrow 3\pi$ with a dispersive approach

Emilie Passemar, IFIC-CSIC, Valencia
emilie.passemar@ific-uv.es

FlaviA net

Laboratoire de Physique Théorique
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Work in collaboration with
G. Colangelo, S. Lanz (ITP-Bern)
Still in progress!

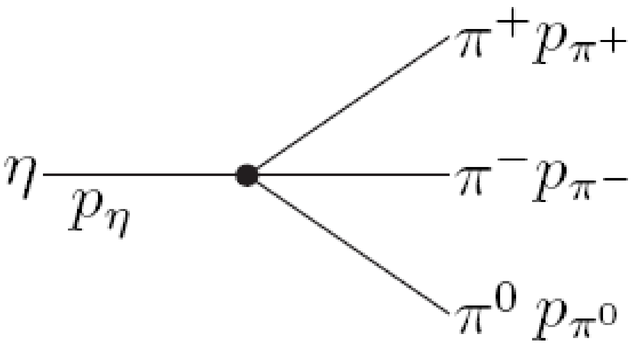
Outline

1. Introduction and Motivation
2. Dispersive approach: the method
3. Preliminary results
4. Conclusion and Outlook

1. Introduction and motivation

1.1 Definitions

- η decay: $\eta \to \pi^+ \pi^- \pi^0$



$$s = \left(p_{\pi^+} + p_{\pi^-}\right)^2, \quad t = \left(p_{\pi^-} + p_{\pi^0}\right)^2$$

$$u = \left(p_{\pi^0} + p_{\pi^+}\right)^2$$

$$s + t + u = M_\eta^2 + M_{\pi^0}^2 + 2M_{\pi^+}^2 \equiv 3s_0$$

$$\langle \pi^+\pi^-\pi^0{}_{out} | \eta \rangle = i(2\pi)^4 \delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right) A(s,t,u)$$

$$\langle \pi^0\pi^0\pi^0{}_{out} | \eta \rangle = i(2\pi)^4 \delta^4\left(p_\eta - p_{\pi^0} - p_{\pi^0} - p_{\pi^0}\right) \bar{A}(s,t,u)$$

- With

$$\bar{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)$$

1.1 Definitions

- Pions in $I = 1$ state ⇨ $A \sim (m_u - m_d)$ or α_{em}

- α_{em} effects are small (but large via $M_{\pi+}$-$M_{\pi-}$), second order

[Sutherland & Bell '68]

$$\mathcal{H}_{\text{QED}}(x) = -\frac{1}{2}e^2 \int dy D^{\mu\nu}(x-y) T(j_\mu(x) j_\nu(y))$$

- Decay via isospin breaking ⇨ access to R or/and Q.

$$\mathcal{H}_{\text{QCD}}(x) = \frac{m_d - m_u}{2}(\bar{d}d - \bar{u}u)(x)$$

$$\Gamma_{\eta\to 3\pi} \propto |A|^2 \propto Q^{-4}$$

$$Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$$ or $$R \equiv \frac{m_s - \hat{m}}{m_d - m_u}$$ with $$\hat{m} \equiv \frac{m_u + m_d}{2}$$

1.2 Why is it so important to measure Q/R ?

- Major input for isospin breaking quantity calculations (π^0-η mixing)

  ⇨ needed for high precision physics

- Examples:
  - K^+_{l3} decays and extraction of V_{us} **[ISTRA+, NA62]**

$$\Gamma_{K^{+/0}l3} = \frac{Br_{K^{+/0}l3}}{\tau_{K^{+/0}}} = \frac{C_K^2 G_F^2 m_{K^{+/0}}^5}{192\pi^3} S_{EW} \left(1 + 2\Delta_{EM}^{K^{+/0}l} + 2\Delta_{SU(2)}\right) \left| f_+^{K^0\pi^-}(0) V_{us} \right|^2 I_{K^{+/0}}^l$$

$$\Delta_{SU(2)} = \frac{f_+^{K^+}(0)}{f_+^{K^0}(0)} - 1$$

$\Delta_{SU(2)}$ related at NLO in ChPT to the quark mass ratio:

$$\Delta_{SU(2)} = \frac{3}{4}\frac{1}{Q^2}\left[\frac{M_K^2}{M_\pi^2} + \frac{\chi_{p^4}}{2}(1+S)\right]$$

Chiral correction

$$\chi_{p^4} = 0.219$$

$$S = \frac{m_s}{\hat{m}}$$

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⇨ Q^2 major input in $\Delta_{SU(2)}$, very important for **NA62** measurements !

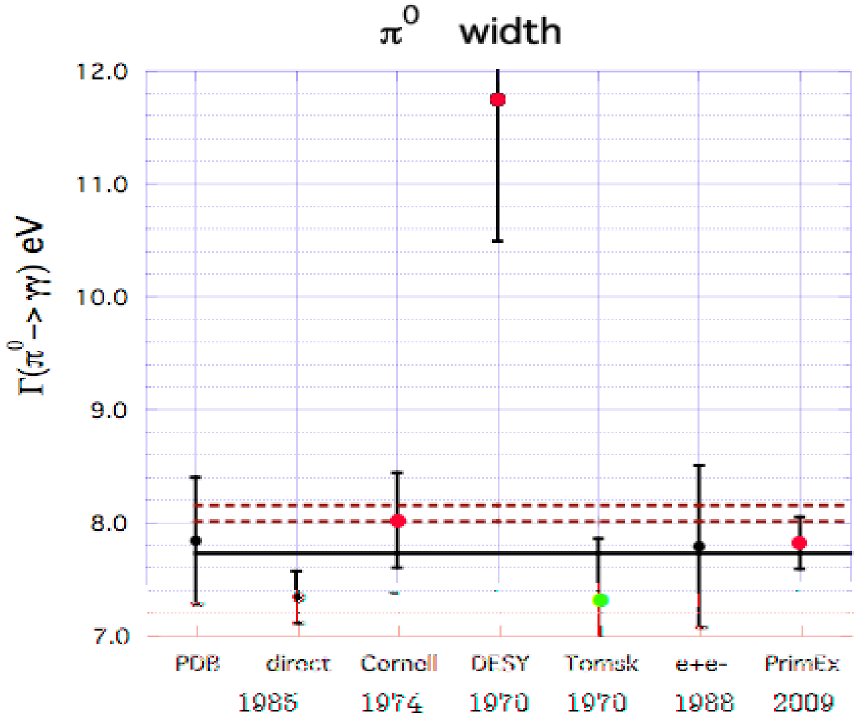
1.2 Why is it so important to measure Q/R ?

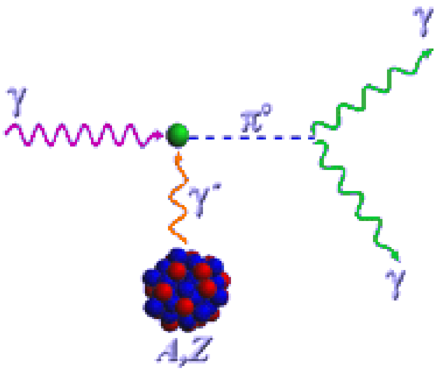
- Examples:
  - Prediction of $\Gamma\left(\pi^0 \to 2\gamma\right)$

[Ananthanarayan & Moussallam'02]
[Goity, Bernstein & Holstein'02]
[Kampf & Moussallam'09]

$$A\left(\pi^0 \to 2\gamma\right) = \frac{\alpha}{\pi F_\pi}\left[1 + \frac{1}{R}(0.93 \pm 0.12) - (0.34 \pm 0.14) \times 10^{-2}\right]$$

⇨ Measurement of F_π independent of the EW couplings of quarks.





[PrimEx]

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- Examples:
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⇒ Measurement of F_π independent of the EW couplings of quarks. **[PrimEx]**

- Test of the standard ChPT : if 3 flavour condensate suppressed due to $s\bar{s}$ condensate fluctuation ⇒ other value for R (Ex: Generalized ChPT)

1.3 How to measure R or Q ?

- From kaon mass splitting formula

$$\Rightarrow \quad Q^2 = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{\left(M_{K^0}^2 - M_{K^+}^2\right)_{QCD}}$$

[Gasser & Leutwyler'84]

Including electromagnetic corrections :

$$Q^2 = \frac{\Delta_{K\pi} M_K^2 \left(1 + \mathcal{O}(m_q^2)\right)}{M_\pi^2 \left[\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0} - (\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0})_{\mathrm{EM}}\right]}$$

- Dashen Theorem : $(\Delta M_K^2)_{EM} = (\Delta M_\pi^2)_{EM} + O(e^2 m_q)$
- Violation of the Dashen Theorem: The corrections can be large due to $e^2 m_s$ corrections: **[Urech'98, Ananthanarayan & Moussallam'04]**

$$(\Delta M_K^2)_{EM} - (\Delta M_\pi^2)_{EM} = e^2 M_K^2 \,(A_1 + A_2 + A_3) + O(e^2 M_\pi^2)$$

$$\Rightarrow \quad (\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0})_{\mathrm{EM}} = -1.5\,\Delta_{\pi^+\pi^0}$$

[Kastner &Neufeld'07]

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$$Q^2 = \frac{\Delta_{K\pi} M_K^2 \left(1 + \mathcal{O}(m_q^2)\right)}{M_\pi^2 \left[\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0} - \left(\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0}\right)_{\rm EM}\right]}$$

- From lattice calculations: N_f=2 [**RBC07**], N_f=2+1 [**MILC09**]

- From a comparison of the measurements K^0_{l3}/K^+_{l3}

- From $\eta \rightarrow 3\pi$ decays

$$\Gamma_{\eta\to 3\pi} = \frac{1}{256\pi^3 M_\eta^3} \int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \, \left|A(s,t,u)\right|^2$$

From experiment

$$A = -\frac{1}{Q^2} \frac{M_K^2}{M_\pi^2} \frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2} M(s,t,u)$$

1.3 How to measure R or Q ?

- From kaon mass splitting formula

$$Q^2 = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{\left(M_{K^0}^2 - M_{K^+}^2\right)_{QCD}}$$

[Gasser&Leutwyler'84]

- From lattice calculations: N_f=2 [**RBC07**], N_f=2+1 [**MILC09**]

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- From $\eta \rightarrow 3\pi$ decays

$$\Gamma_{\eta\rightarrow 3\pi} = \frac{1}{256\pi^3 M_\eta^3}\int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \, |A(s,t,u)|^2$$

From experiment

$$A = -\frac{\left(M_\eta^2 - M_\pi^2\right)}{4\sqrt{3}F_\pi^2}\frac{1}{R}M(s,t,u)$$

⇨ Calculation of M in the following

1.4 Study of $\eta \to 3\pi$

- Different approaches
  - Chiral Perturbation Theory **[Gasser&Leutwyler'84, Bijnens&Ghorbani'06]**

    ⇨ converges slowly.
    - Tree level: $\Gamma_{\eta\to 3\pi} = 66 \text{ eV}$ **[Cronin'67, Osborn&Wallace'70]**
    - One loop: $\Gamma_{\eta\to 3\pi} = 160 \text{ eV}$ **[Gasser&Leutwyler'84]**
    - Experimentally: $\Gamma_{\eta\to 3\pi} = 295 \text{ eV}$ **[PDG'08]**

  Large enhancement between tree level and one loop due to $\pi\pi$ rescattering effects (anticipated by **[Roiesnel & Truong'81]**)

  - Dispersive approach: use of analyticity, unitarity and crossing symmetry

    ⇨ M **[Kambor,Wiesendanger&Wyler'96, Anisovich&Leutwyler'96]**

    → iterative procedure

    → take into account **all** the $\pi\pi$ rescattering effects

1.4 Study of $\eta \to 3\pi$

- Different approaches
  - Chiral Perturbation Theory **[Gasser&Leutwyler'84, Bijnens&Ghorbani'06]**
  - Unitarized Chiral Perturbation Theory **[Borasoy & Nissler'96, Beisert & Borasoy'03]**
  - Dispersive approach **[Kambor,Wiesendanger&Wyler'96, Anisovich&Leutwyler'96]**

- New dispersive analysis
  - new inputs available: $\pi\pi$ phase shifts

    [Ananthanarayan et al'01, Colangelo et al'01, Descotes-Genon et al'01, Garcia-Martin et al'09]
  - Recent and forthcoming measurements [**KLOE, CBall, WASA**]
  - Possible Improvements: electromagnetic effects, inelasticity…

1.4 Study of η → 3π

- In the following ⇒ calculation of M, charged channel

$$\Gamma_{\eta\to\pi^+\pi^-\pi^0} = \frac{1}{256\pi^3 M_\eta^3}\int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \, \left|A(s,t,u)\right|^2$$

$$A = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2} M(s,t,u)$$

- NB: Other approaches:
  - Analytical dispersive **[Kampf, Knecht, Novotný, Zhadral '09]**
  - Use of resummed ChPT **[Descotes-Genon, Kolesár, Novotný in progress]**

2. Dispersive approach: the Method

[Anisovich & Leutwyler'96]

2.1 Decomposition of the amplitude

- Much easier to write dispersion relations for functions of 1 variable !

- Decomposition of the amplitude as a function of isospin states

$$M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)$$

[Fuchs, Sazdjian & Stern'93]

  - M_I isospin I rescattering in two particles
  - Amplitude in terms of S and P waves, $disc\ t_l^I(s) = 0$ for $l \geq 2$

    ⇨ exact up to NNLO ($\mathcal{O}(p^6)$)
  - Main two body rescattering corrections inside M_I

- Write dispersion relations for each M_I, functions of one variable with only a right hand cut ⇨ $disc\ M_I = disc\ t_l^I(s)$ $(s > 4M_\pi^2)$

- $t_l^I(s) = M_I(s) + \hat{M}_I(s)$ with $\hat{M}_I$ singularities in the t and u channels

2.3 Dispersion relations for the M_I

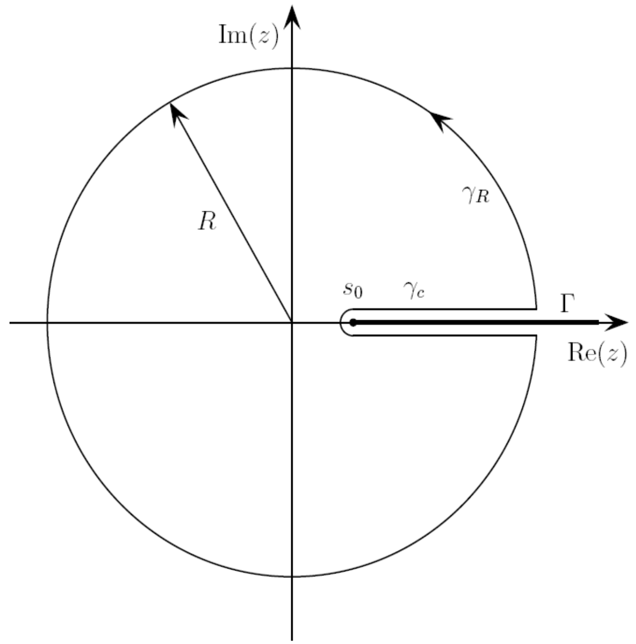
- 3 body dispersive is difficult ⇨ keep only 2 body cut

  Process $\pi\eta \to \pi\pi \quad \left(M_\eta^2 < 3M_\pi^2\right)$

  Standard dispersive analysis and analytical continuation to physical M_η^2

- Relation of dispersion:

  Cauchy Theorem : $$M_I(z) = \frac{1}{2i\pi}\oint \frac{M_I(\xi)}{\xi - z}d\xi$$



$$M_I(z) = \frac{1}{2i\pi}\left(\int_{4M_\pi^2}^{\Lambda^2} \frac{M_I(s+i\varepsilon)}{s-z}ds + \oint_{|s|=\Lambda^2} \frac{M_I(s)}{s-z}ds + \int_{\Lambda^2}^{4M_\pi^2} \frac{M_I(s-i\varepsilon)}{s-z}ds\right)$$

⇨ $$M_I(z) = \frac{1}{2i\pi}\left(\int_{4M_\pi^2}^{\infty} \frac{M_I(s+i\varepsilon) - M_I(s-i\varepsilon)}{s-z}ds\right)$$

⇨ $$M_I(s) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{discM_I(s')}{s'-s-i\varepsilon}ds'$$

- M_I can be reconstructed everywhere from the knowledge of disc M_I

2.3 Dispersion relations for the M_I

- **disc** M_I has to be known along the cut

- Elastic unitarity :

$$\Rightarrow \quad disc\ M_I = disc\ t_l^I(s) = \theta\left(s - 4M_\pi^2\right) t_l^I(s) \sin\delta_I(s) e^{-i\delta_I(s)}$$

$$\left(t_l^I = M_I(s) + \hat{M}_I(s)\right) \qquad \pi\pi \text{ phase shift}$$

- 3 functions related to each others $\Rightarrow$ $\hat{M}_I$

- Number of subtractions n given by the asymptotic behaviour of M_I

$\Rightarrow$ Froissard bound: $M(s,t,u) \overset{s,t,u \to \infty}{=} O(s,t,u)$

2.3 Dispersion relations for the M_I

- Unitarity $\Rightarrow$ $$disc\ M_I = \left(M_I(s) + \hat{M}_I(s)\right) \sin\delta_I(s) e^{-i\delta_I(s)} \theta\left(s - 4M_\pi^2\right)$$

- Write dispersion relations for each M_I
  - NB: Froissart bound for the asymptotic behaviour $M(s,t,u) \overset{s,t,u\to\infty}{=} O(s,t,u)$

$$M_0(s) = a_0 + b_0 s + c_0 s^2 + \frac{s^3}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^3} \frac{disc\ M_0(s)}{s' - s - i\varepsilon}$$

$$M_1(s) = a_1 + b_1 s + \frac{s^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{disc\ M_1(s)}{s' - s - i\varepsilon}$$

$$M_2(s) = a_2 + b_2 s + c_2 s^2 + \frac{s^3}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^3} \frac{disc\ M_2(s)}{s' - s - i\varepsilon}$$

- But solutions not uniquely defined !

2.4 Ambiguity in the standard dispersion relations

- To see it, consider the solution of the Omnès problem $\left(\hat{M}_I(s')=0\right)$
- Unitarity ⇨ $disc\ M_I = M_I(s)\sin\delta_I(s)e^{-i\delta_I(s)}\theta\left(s-4M_\pi^2\right)$

⇨ Solution: $M_I(s)=m_I(s)\ \Omega_I(s)$ with $\Omega_I(s)=\exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)$

polynomial

$\left(\text{NB: }\Omega(s)\rightarrow\left(\frac{\Lambda^2-s}{\Lambda^2}\right)^{-\alpha}\right)$

- Dispersion relation for M_I:

$$M_I(s)=P_I(s)+\frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_I(s')e^{-i\delta_I(s')}}{s'-s-i\varepsilon}M_I(s')$$

- Asymptotic behaviour : $M_I(s)\xrightarrow{s\to\infty}P_I(s)$

  If P_I of degree n-1 then m_I (I=0 or 1) is of degree n $\left(\delta_0(s),\delta_1(s)\rightarrow\pi\right)$

  ⇨ M_I admits a one parameter family of solutions.

2.5 Unambigous dispersion relations for the M_I's

- Write instead dispersion relations for $M_I(s)/\Omega_I(s)$

$$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0 \hat{M}_0(s)}{|\Omega_0(s')|(s'-s-i\varepsilon)}\right)$$

Similar for the others ⇨ Coupled equations

- Unitarity ⇨

$$disc\ \frac{M_I(s)}{\Omega_I(s)} = \frac{\sin\delta_I(s)\hat{M}_I(s)}{|\Omega_I(s)|}\theta\left(s-4M_\pi^2\right)$$

- $\Omega_I(s)$: Omnès function

⇨ $$\Omega_I(s) = \left(M_I \text{ with } \hat{M}_I = 0\right) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)$$

2.5 Unambigous dispersion relations for the M_I's

- $\delta_0(s), \delta_1(s) \overset{s\to\infty}{\longrightarrow} \pi, \; \delta_2(s) \to 0$

- Explicitely, 3 coupled equations to solve

$$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0(s')\,\hat{M}_0(s')}{|\Omega_0(s')|(s'-s-i\varepsilon)}\right)$$

$$M_1(s) = \Omega_1(s)\left(\alpha_1 + \beta_1 s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1(s')\,\hat{M}_1(s')}{|\Omega_1(s')|(s'-s-i\varepsilon)}\right)$$

$$M_2(s) = \Omega_2(s)\left(\alpha_2 + \beta_2 s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_2(s')\,\hat{M}_2(s')}{|\Omega_2(s')|(s'-s-i\varepsilon)}\right)$$

- 7 “unknown” constants to be determined

2.6 Resolution

$$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0 \hat{M}_0(s)}{|\Omega_0(s')|(s'-s-i\varepsilon)}\right)$$

- Start from tree level → $\hat{M}_I$ + one inputs the ππ phase shifts δ_I

[Watson's theorem]

Tree level: $M(s,t,u) = T(s) = 1 + 3\frac{s-s_0}{M_\eta^2 - M_\pi^2}$

– $T(s=s_0)=1$ center of the Dalitz plot $\left(3s_0 \equiv s+t+u = M_\eta^2 + M_{\pi^0}^2 + 2M_{\pi^+}^2\right)$

⇨ $M_0^{\text{tree}}(s) = T(s),\ M_1^{\text{tree}}(s) = M_2^{\text{tree}}(s) = 0$

Computation of the $\hat{M}_I$s

$$\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle(s) + 2\left(s - s_0\right)\langle M_1\rangle(s) + \frac{20}{9}\langle M_2\rangle(s) + \frac{2}{3}\kappa(s)\langle zM_1\rangle(s)$$

- Similar relations for the others $\hat{M}_I$s

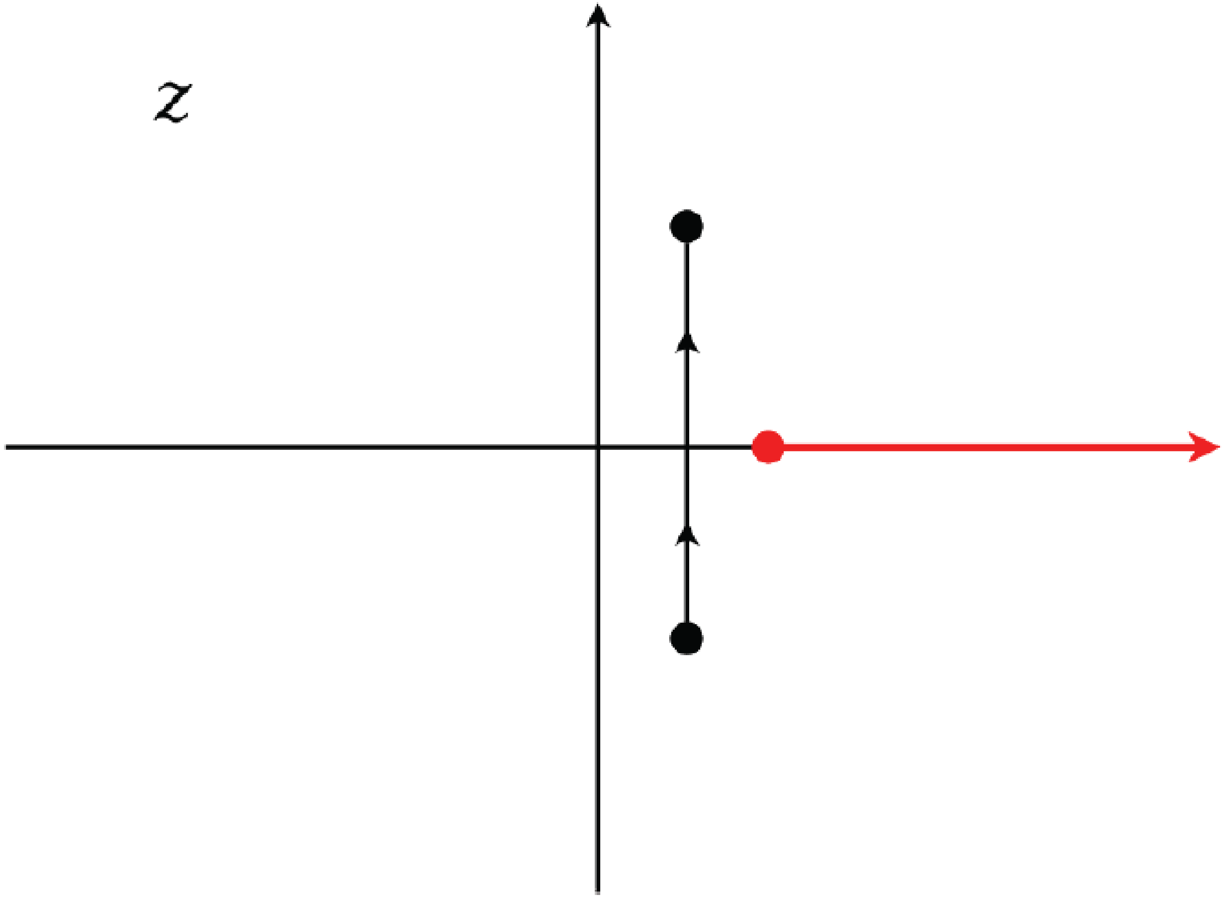
- $\kappa(s) = \sqrt{\frac{s - 4M_\pi^2}{s}}\sqrt{\left(\left(M_\eta + M_\pi\right)^2 - s\right)\left(\left(M_\eta - M_\pi\right)^2 - s\right)}$

- Angular averages $\langle z^n f\rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\ z^n f\left(\frac{3s_0 - s + z\kappa(s)}{2}\right)$

with $z = \cos\theta$ scattering angle

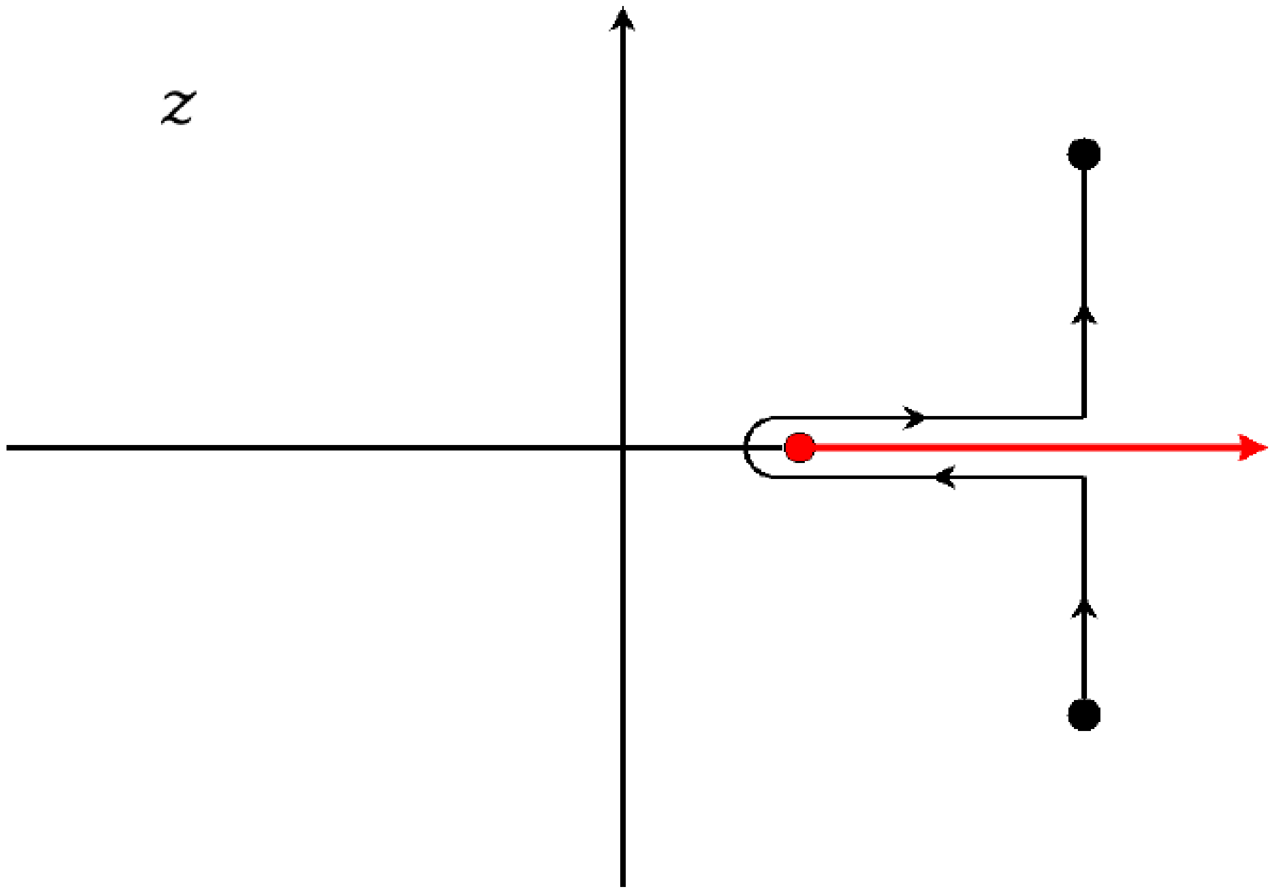
Computation of the $\hat{M}_I$s

- Integration path



Computation of the $\hat{M}_I s$

- Integration path



2.6 Iterative solution

$$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0 \hat{M}_0(s)}{|\Omega_0(s')|\left(s'-s-i\varepsilon\right)}\right)$$

- Start from tree level → $\hat{M}_I$ + one inputs the ππ phase shifts δ_I

  [Watson's theorem]

- Determine the M_Is and then M
  - Calculate the dispersive integral
  - Determine the subtraction constants: ⇨ Matching with ChPT at one loop (2 sum rules + Adler zero)

Subtraction constants, 4 combinations in M

⇨ 4 subtraction constants to be determined α_0, β_0, γ_0 and β_1 (α_1, α_2, β_2, γ_2=0)

Determination of the subtraction constants

- Decomposition of the amplitude not unique :

$$M_0(s) \to M_0(s) + 3c_1\left(s - s_0\right) + \frac{4}{3}c_2 + c_3\left(3s_0 - \frac{5}{3}s\right)$$

$$M_1(s) \to M_1(s) + c_1$$

$$M_2(s) \to M_2(s) + c_2 + c_3 s$$

⇨ M remains invariant

⇨ $\alpha_1 = \alpha_2 = \beta_2 = 0$

Determination of the subtraction constants

- Dispersive integrals to solve

$$M_0(s)=\Omega_0(s)\left(\alpha_0+\beta_0 s+\gamma_0 s^2+\frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0(s')\,\hat{M}_0(s')}{|\Omega_0(s')|(s'-s-i\varepsilon)}\right)$$

$$M_1(s)=\Omega_1(s)\left(\beta_1 s+\frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1(s')\,\hat{M}_1(s')}{|\Omega_1(s')|(s'-s-i\varepsilon)}\right)$$

$$M_2(s)=\Omega_2(s)\frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_2(s')\,\hat{M}_2(s')}{|\Omega_2(s')|(s'-s-i\varepsilon)}$$

⇨ 4 subtraction constants to be determined from a matching to the one loop ChPT calculations

Determination of the subtraction constants

- β_1 and γ_0 determined from sum rules due to the aymptotic behaviour of $M_0 + 4/3\, M_2$ and $s\, M_1 + M_2$

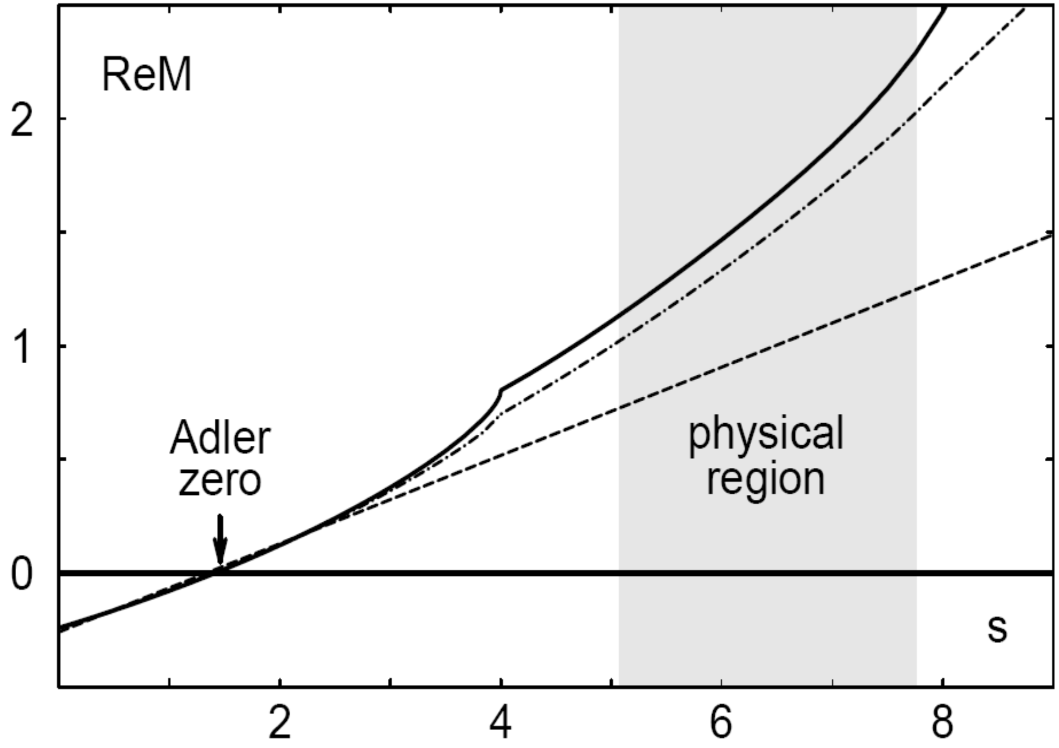
$$\left[M_1^{'}(0) + \frac{1}{2} M_2^{''}(0) \right]_{disp} = \left[M_1^{'}(0) + \frac{1}{2} M_2^{''}(0) \right]_{one\ loop} \Longleftrightarrow \beta_1 \approx -\frac{4L_3 - 1/64\pi^2}{\left(M_\eta^2 - M_\pi^2 \right) F_\pi^2}$$

$$\left[M_0^{''}(0) + \frac{4}{3} M_2^{''}(0) \right]_{disp} = \left[M_0^{''}(0) + \frac{4}{3} M_2^{''}(0) \right]_{one\ loop} \Longleftrightarrow \gamma_0 \approx 0$$

- α_0 and β_0 determined from a matching to an Adler zero and its slope

Determination of the subtraction constants

- α_0 and β_0 determined from a matching to an Adler zero and its slope
  ⇨ protected from large corrections by SU(2) x SU(2) symmetry



2.6 Iterative solution

- Start from 1rst iteration result → $\hat{M}_I$
- Determine the M_Is and then M
  - Calculate the dispersive integral
  - Determine the subtraction constants: ⇨ Matching with ChPT at one loop

1 iteration

- Repeat the procedure until convergence

Future determination of the subtraction constants

- Determine them from the KLOE data

- For the moment, fit of $M(s,t,u;\alpha_0,\beta_0,\gamma_0,\beta_1)$ to the Dalitz plot fit of KLOE

$$f(X,Y) = N\left(1 + aY + bY^2 + cX + dX^2 \ldots\right)$$

$$X = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$

$$Y = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

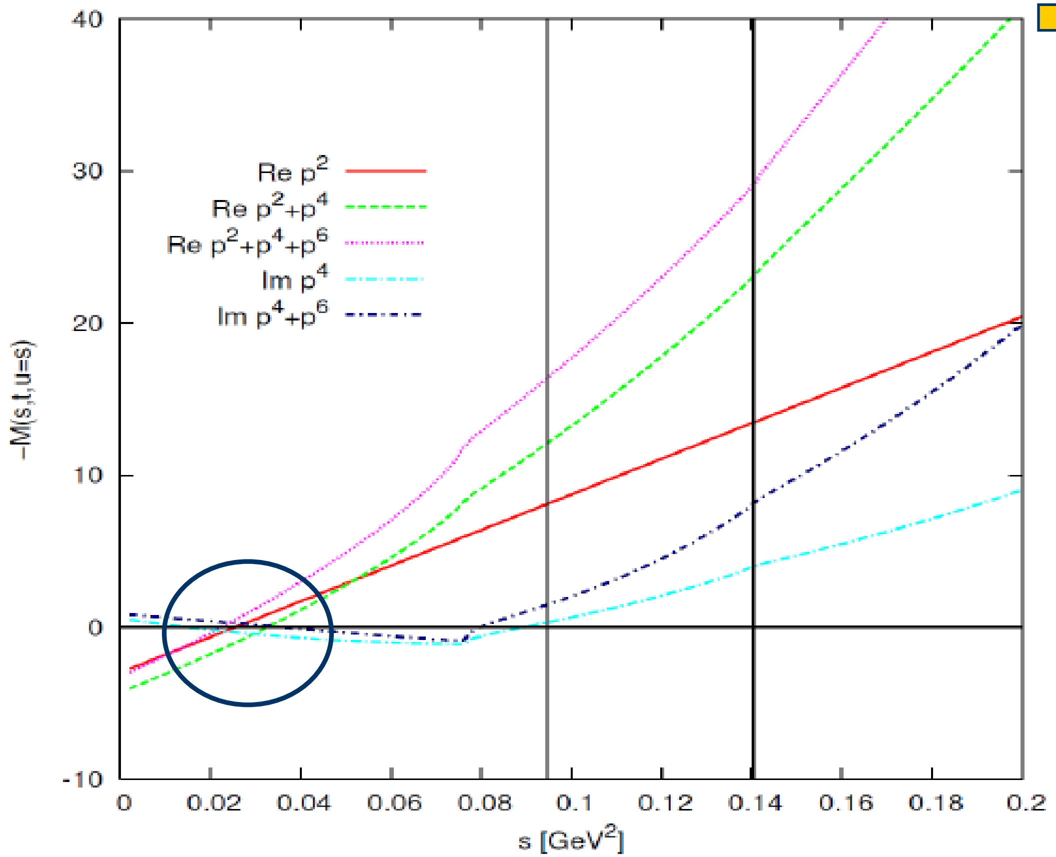
$$Q_c \equiv M_\eta - 2M_{\pi^+} - M_{\pi^0}$$

- The normalization of $f(X,Y)$ is not measured ⇨ factor out α_0 (all constraints on M_I linear)

⇨ $M(s,t,u) = \alpha_0 \tilde{M}\left(s,t,u;\frac{\beta_0}{\alpha_0},\frac{\gamma_0}{\alpha_0},\frac{\beta_1}{\alpha_0}\right)$ ⇨ Fit $\tilde{M}$

Future determination of the subtraction constants

- Determine them from the KLOE data
- We will use ChPT to determine the overall normalization: α_0



⇨ slope of the Adler zero

Region where only moderate change order by order

[Bijnens & Ghorbani'07]

Future determination of the subtraction constants

- Determine them from the KLOE data

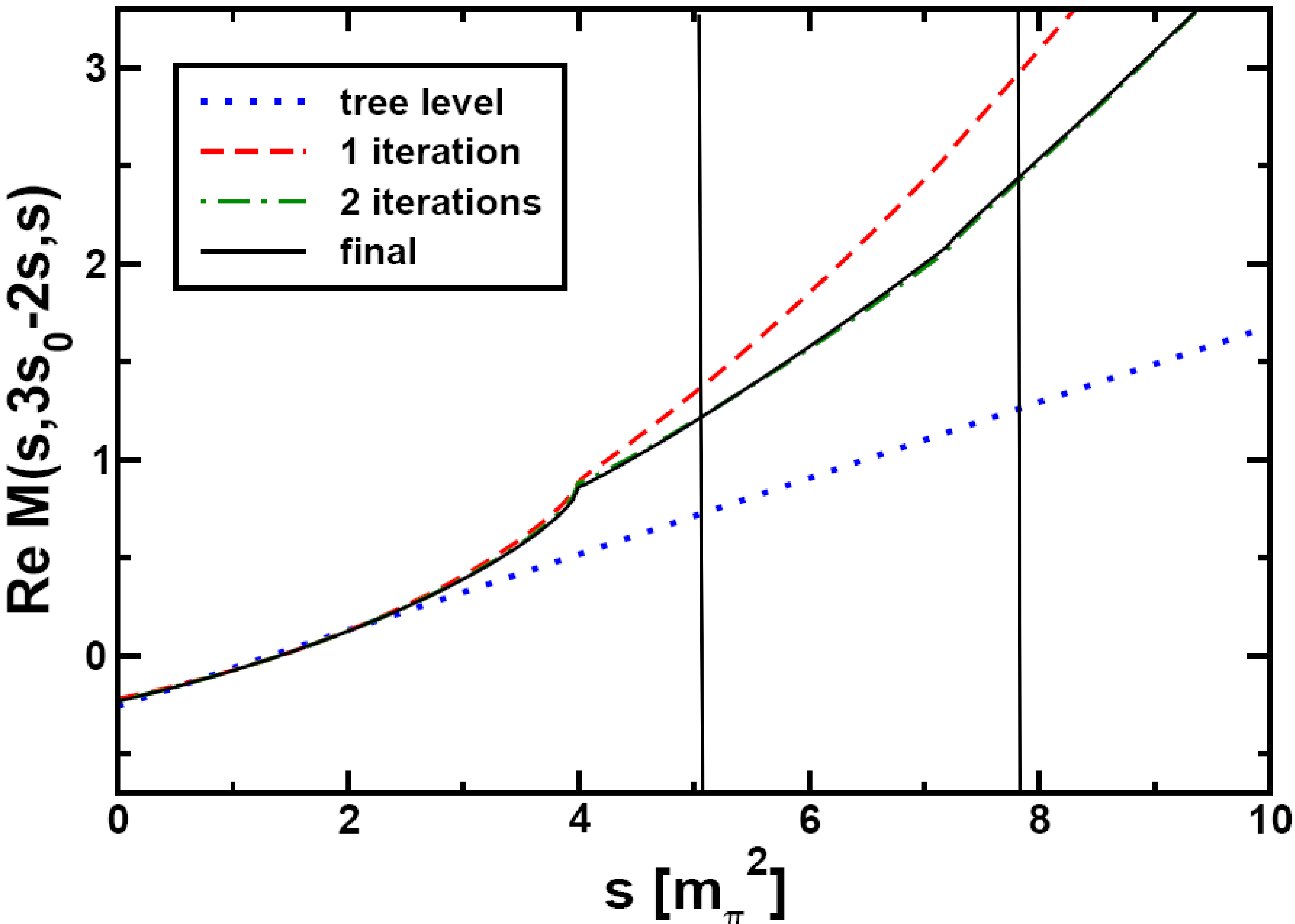
- For the moment fit to a fit
  - χ^2 not uniquely defined
  - No statistical interpretation

  ⇨ Use directly real data

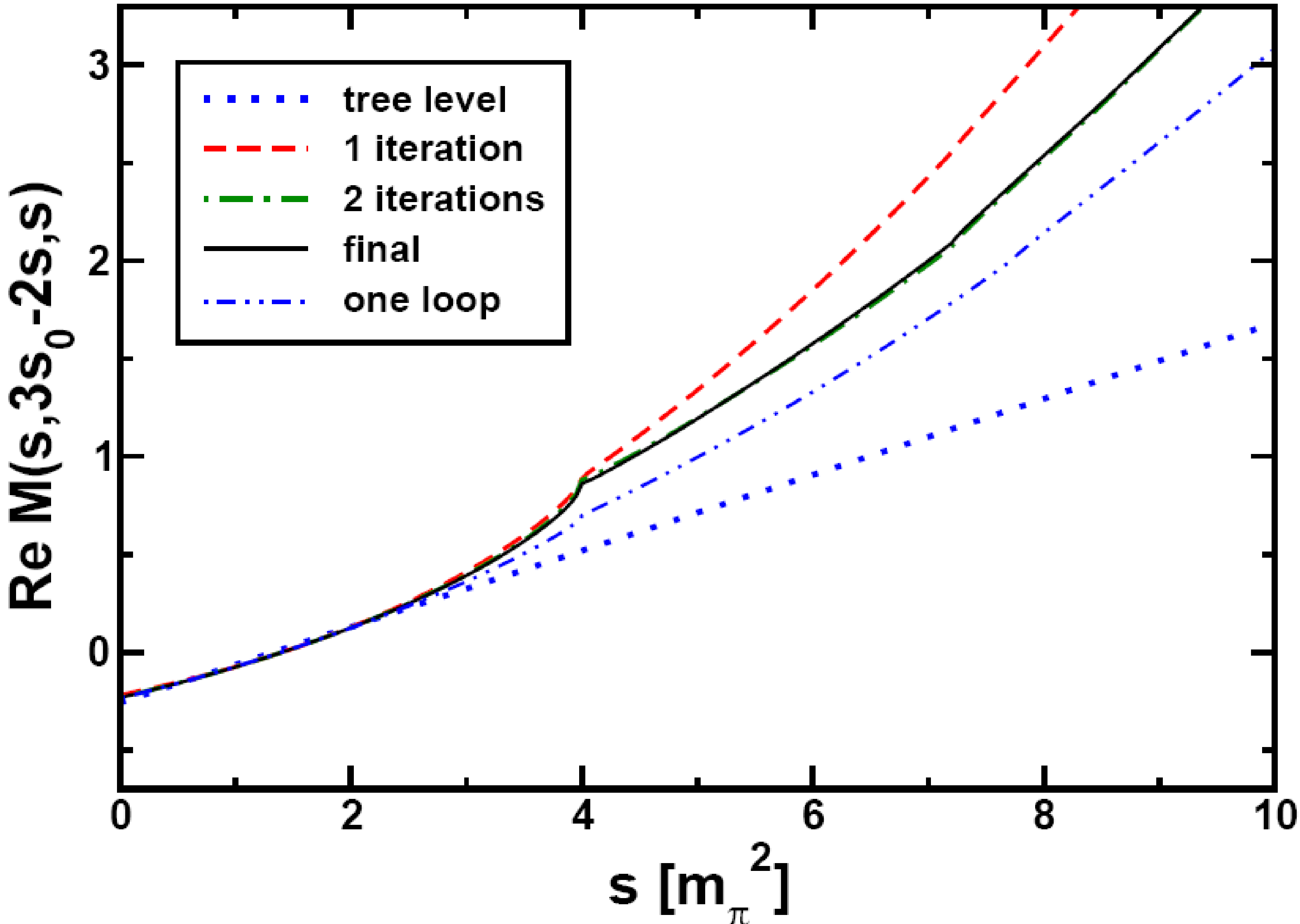
- Results in the following with matching to one loop ChPT

3. Preliminary results

3.1 Results for M(s,t,u) along s=u



3.1 Results for M(s,t,u) along s=u



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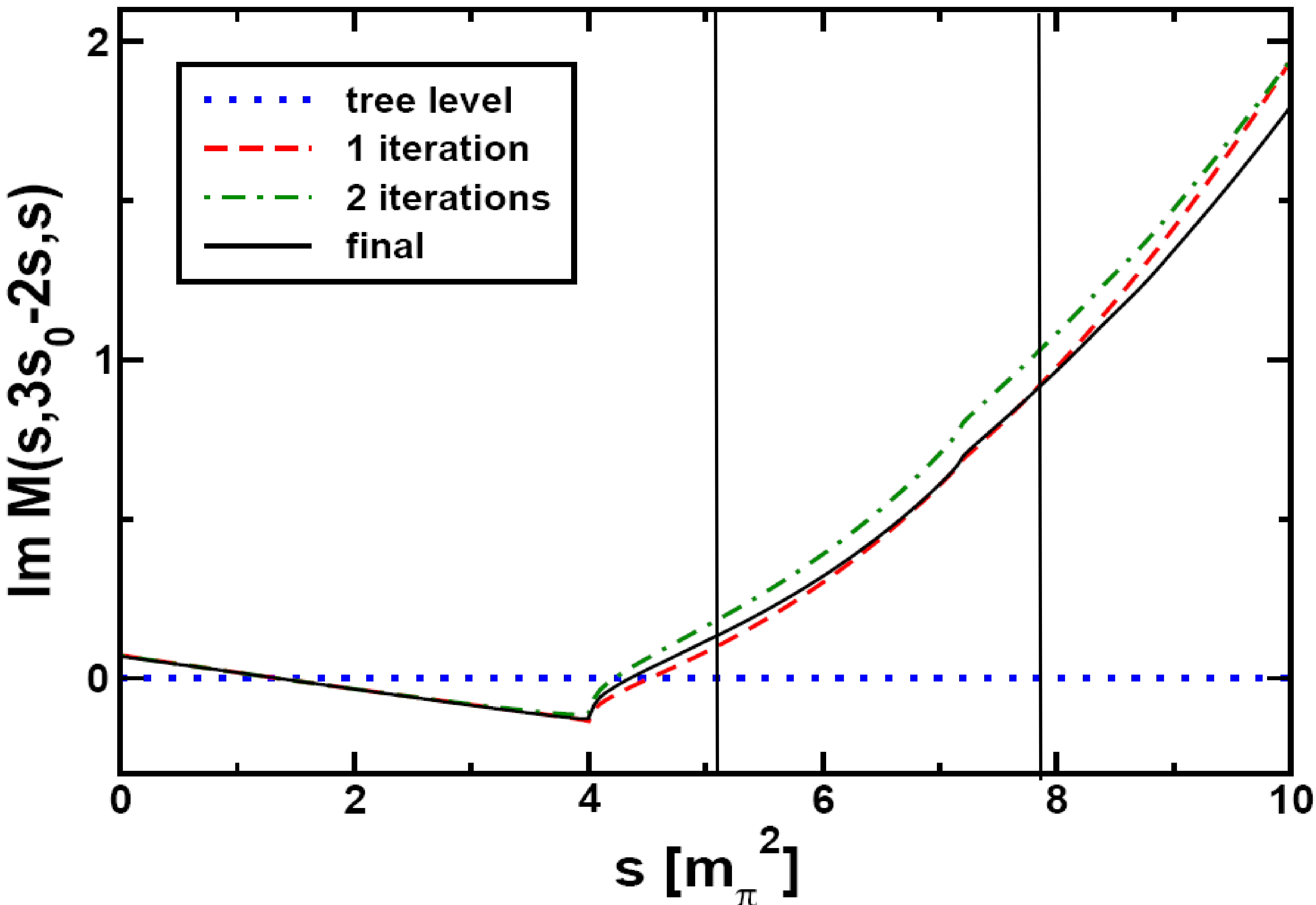
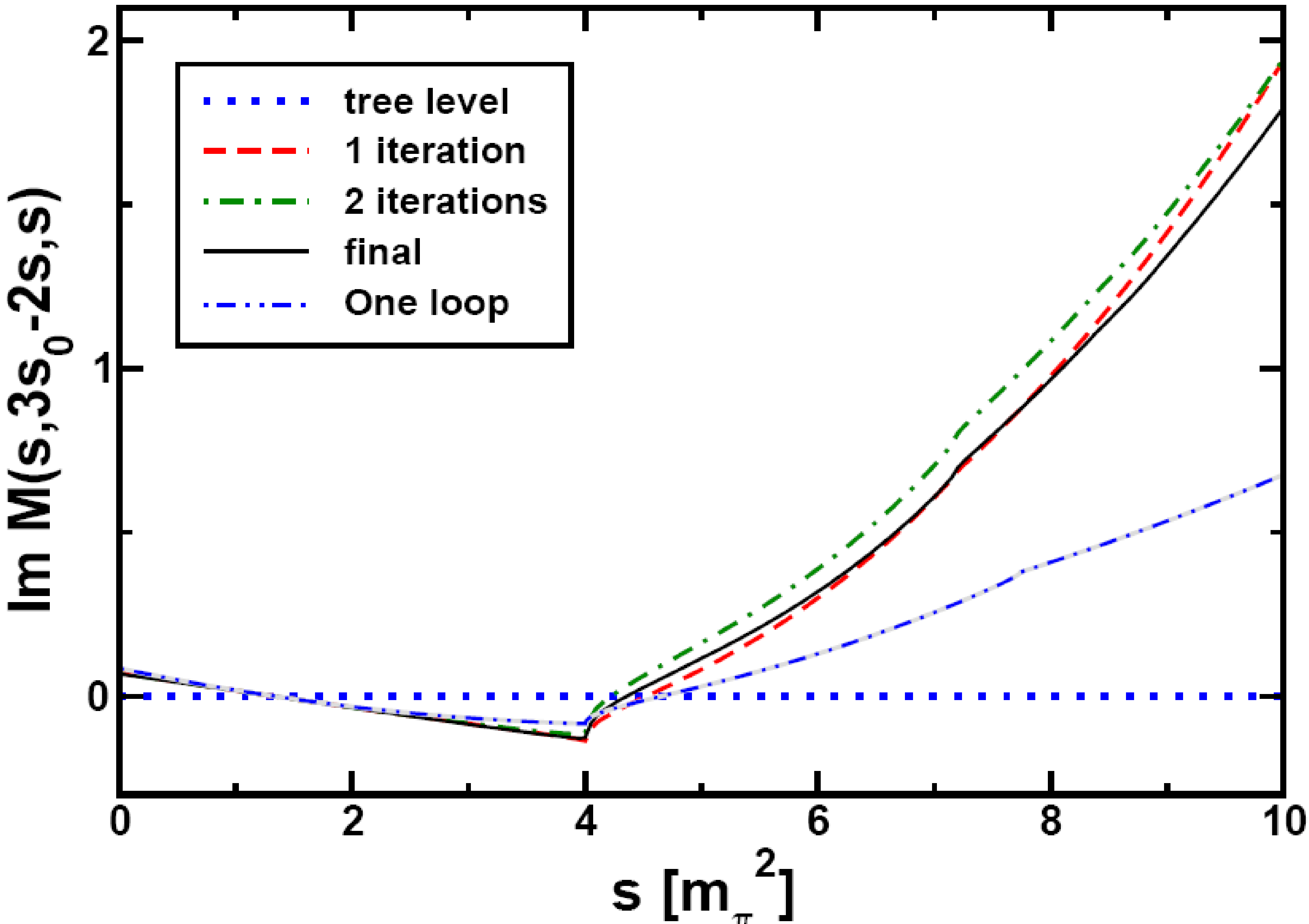
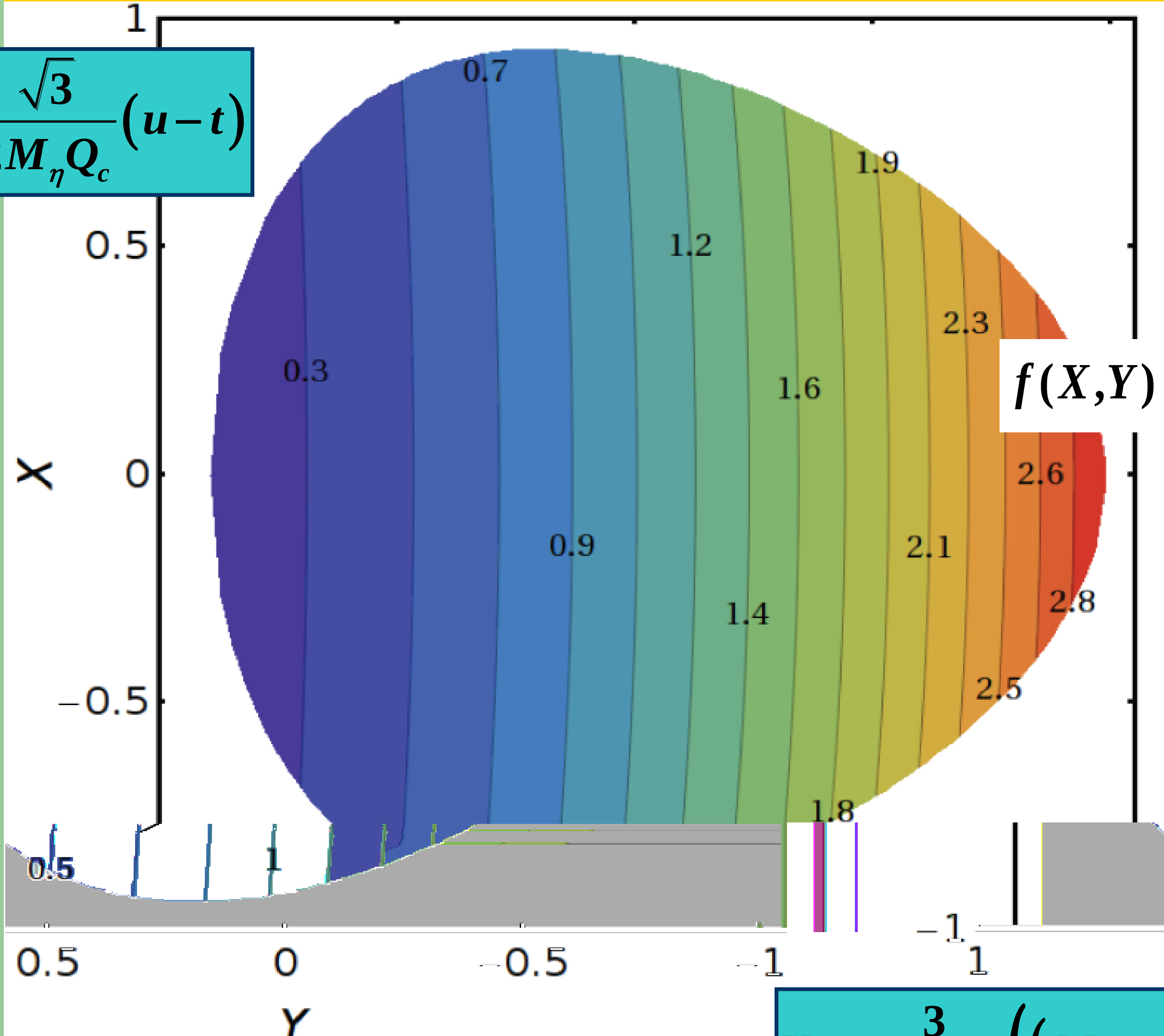


3.1 Results for M(s,t,u) along s=u



3.2 Dalitz plot for $\eta \rightarrow \pi^+ \pi^- \pi^0$

$$X = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$



Measured by **KLOE**

[KLOE'08]

$$f(X,Y) = N\left(1 + aY + bY^2 + cX + \ldots\right)$$

$$Q_c \equiv M_\eta - 2M_{\pi^+} - M_{\pi^0}$$

$$Y = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

3.3 Result for Q

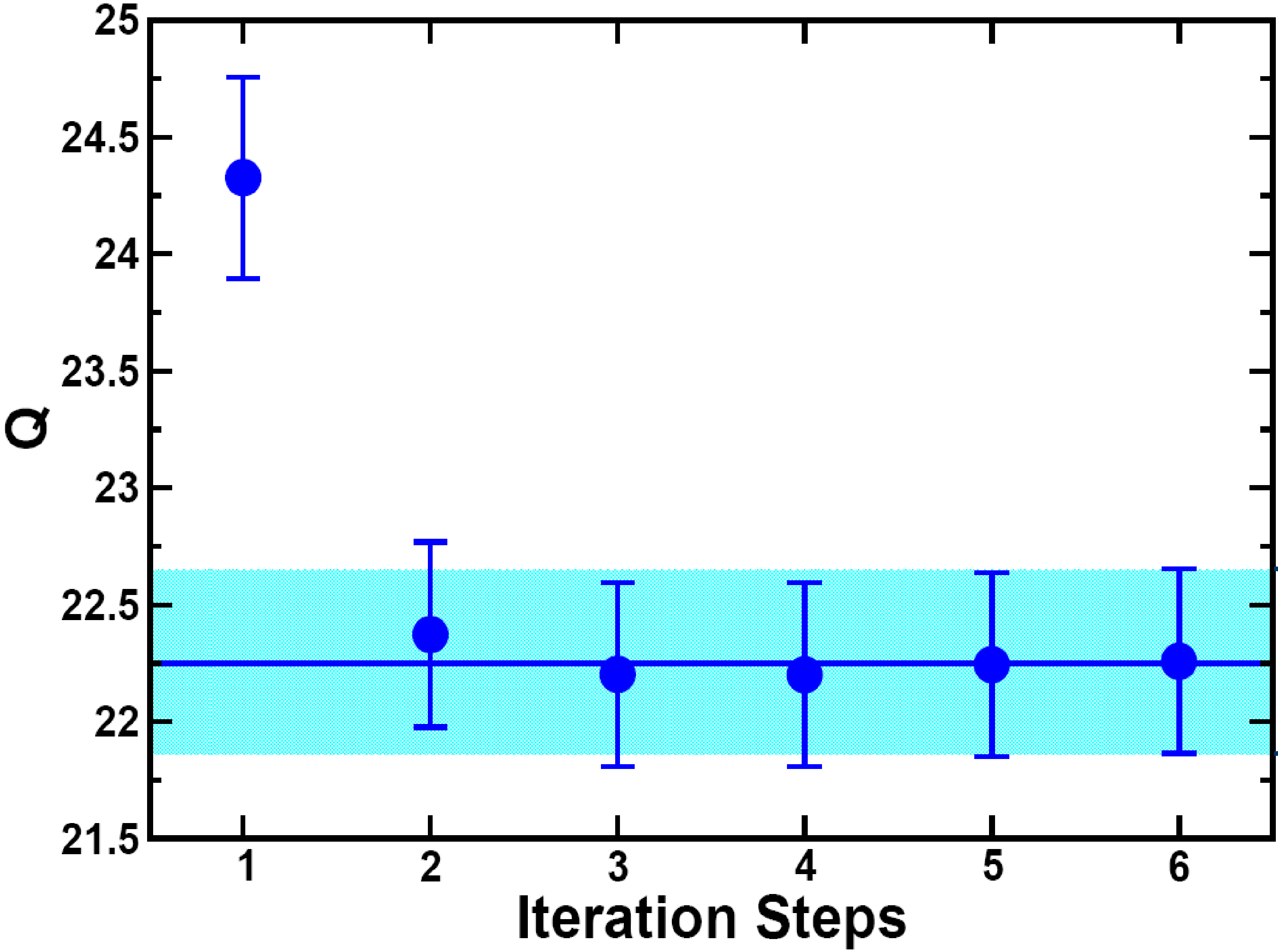
$$\Gamma_{\eta\to 3\pi} = \frac{1}{256\pi^3 M_\eta^3}\int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \, \left|A(s,t,u)\right|^2$$

$\Gamma_{\eta\to 3\pi} = 295 \pm 20 \text{ eV}$ **[PDG'08]**

$$A = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2} M(s,t,u)$$

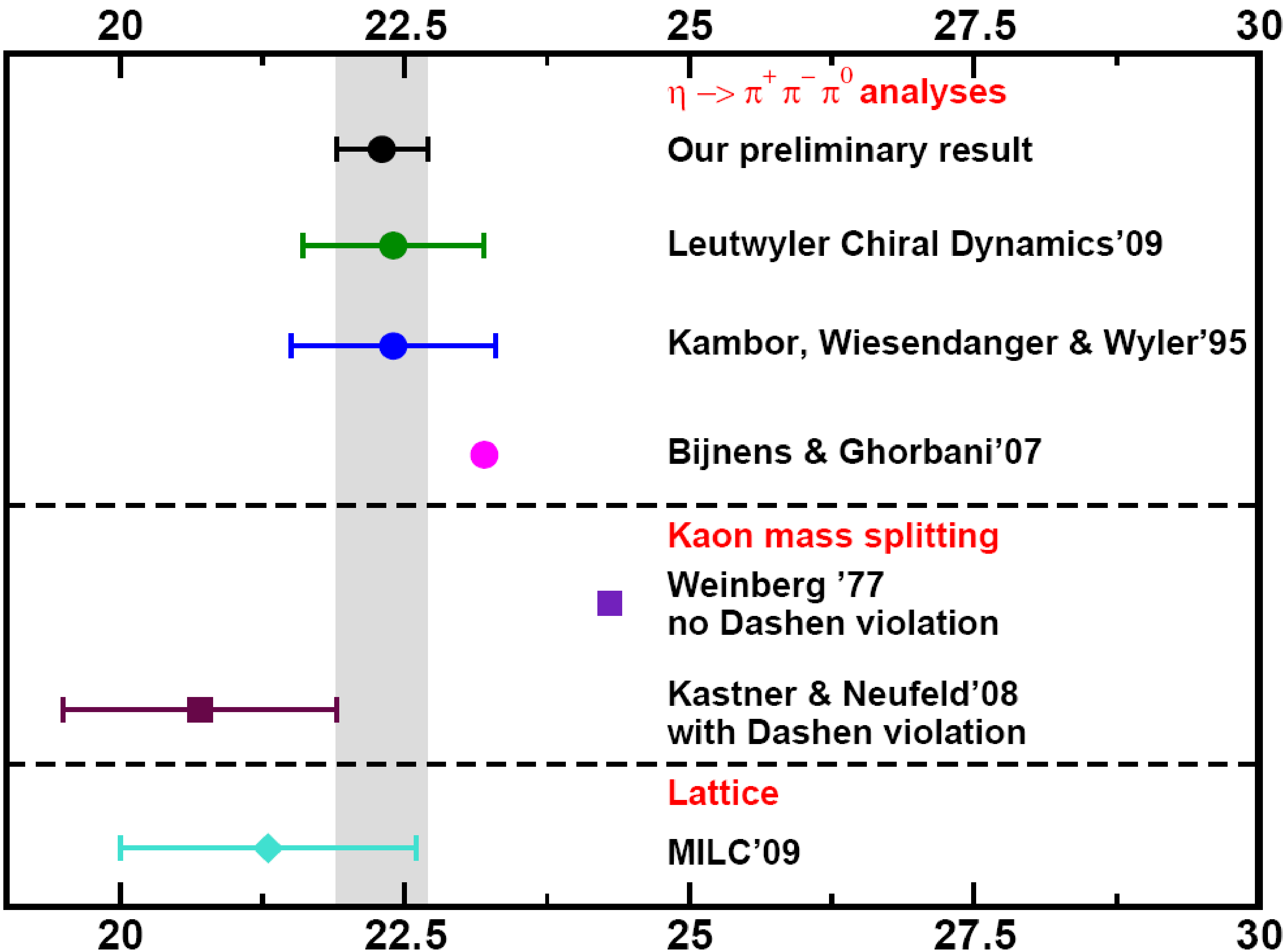
- Preliminary result : ⇒ $Q = 22.3 \pm 0.4$
- Uncertainties from $\Gamma_{\eta\to 3\pi}$ only !
- $R = 34.9 \pm 1.2$ ⇒ $\frac{m_s}{\hat{m}} = 28$

3.3 Result for Q



Preliminary result with exp. uncertainty only !

3.4 Comparison with the other results for Q



3.4 Comparison with the other results for Q

- Q major semi axis of Leutwyler's ellipse

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$

- At leading order:

$$\frac{m_u}{m_d} \stackrel{\text{LO}}{=} \frac{M^2_{K^+} - M^2_{K^0} + 2M^2_{\pi^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} = 0.56\,,$$

$$\frac{m_s}{m_d} \stackrel{\text{LO}}{=} \frac{M^2_{K^+} + M^2_{K^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} = 20.2$$

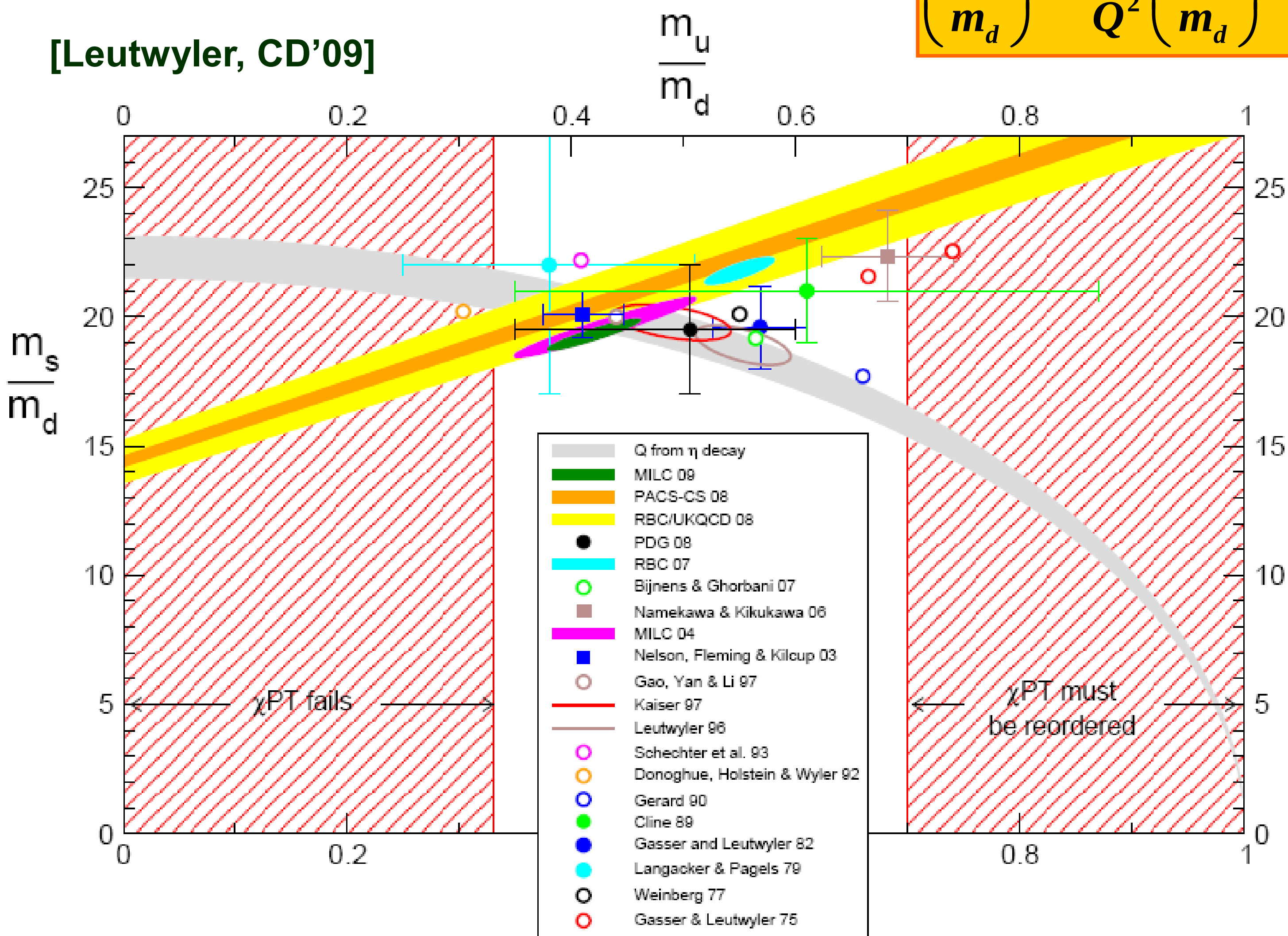
[Weinberg'77]

3.4 Comparison with the other results for Q

- Q major semi axis of Leutwyler's ellipse

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$

[Leutwyler, CD'09]



4. Conclusion and Outlook

4.1 Conclusion

- New dispersive analysis of $\eta \to 3\pi$:
  - New inputs available: $\pi\pi$ phase shifts
  - Recent experimental activity: KLOE, WASA, COSY
  - Theoretical activities:
    - $\eta \to 3\pi$ at two loops **[Bijnens & Ghorbani'07]**
    - $\eta \to 3\pi$ using dispersive techniques, resummed ChPT **[Kampf, Knecht, Novotný, Zhadral in progress]**
    - $\eta \to 3\pi$ using resummed ChPT **[Descotes-Genon, Kolesár, Novotný in progress]**
    - Electromagnetic corrections **[Ditsche, Kubis & Meissner'08]**

- Preliminary result for Q for the charged channel but more quantities to evaluate:
  - branching ratio: $r = \Gamma_{\eta\to\pi^0\pi^0\pi^0} / \Gamma_{\eta\to\pi^+\pi^-\pi^0}$
  - slope parameter α for the neutral channel…

4.2 Outlook:

- Use the experimental data to determine the subtraction constants

  [KLOE'07, CBall'08, WASA'08]

- Study the effects due to $M_{\pi+}$- $M_{\pi 0}$ **[Kupsc, Rusetsky & Gullstrom'08, Kubis & Schneider'09]**

- Analysis of the uncertainties

- Future extensions:
  - Inelasticity effects in the phase shifts
  - Electromagnetic corrections **[Ditsche, Kubis & Meissner'08]**
  - Study of the effects of the discontinuities of D and higher waves in the decomposition of M

Additional Slides

Determination of the subtraction constants

- Asymptotic behaviour of the phase shifts:

$$\delta_0(s),\delta_1(s)\xrightarrow{s\to\infty}\pi,\ \delta_2(s)\to 0 \quad\Rightarrow\quad \begin{aligned}P_0(s)&=\alpha_0+\beta_0 s+\gamma_0 s^2\\ P_1(s)&=\alpha_1+\beta_1 s\\ P_2(s)&=\alpha_2+\beta_2 s\end{aligned} \quad\Rightarrow\quad \boxed{\gamma_2\equiv 0}$$

- Decomposition of the amplitude not unique :

$$M_0(s)\to M_0(s)+3c_1\left(s-s_0\right)+\frac{4}{3}c_2+c_3\left(3s_0-\frac{5}{3}s\right)$$

$$M_1(s)\to M_1(s)+c_1$$

$$M_2(s)\to M_2(s)+c_2+c_3 s$$

⇒ M remains invariant

$$\Rightarrow\quad \boxed{\alpha_1=\alpha_2=\beta_2=0}$$

2.2 Principles of a dispersion relation

- Consider a function f
  - has a branch cut along the real axis starting at s_0
  - Is real for $s \leq s_0$
  - Is analytic in the complex plane

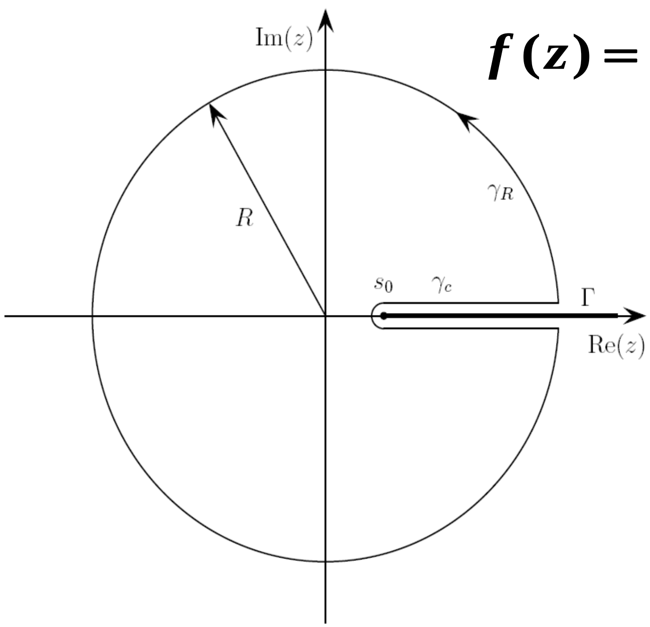
- Schwartz's reflection principle : $f(z^*) = f^*(z)$

$$\Rightarrow \quad f(s+i\varepsilon) - f(s-i\varepsilon) = 2i \,\mathrm{Im}\, f(s+i\varepsilon)$$

- Cauchy Theorem : $f(z) = \dfrac{1}{2i\pi} \oint \dfrac{f(\xi)}{\xi - z} d\xi$

2.2 Principles of a dispersion relation

- Cauchy Theorem : $f(z) = \frac{1}{2i\pi}\oint \frac{f(\xi)}{\xi - z} d\xi$



$$f(z) = \frac{1}{2i\pi}\left(\int_{s_0}^{\Lambda^2} \frac{f(s+i\varepsilon)}{s-z} ds + \oint_{|s|=\Lambda^2} \frac{f(s)}{s-z} ds + \int_{\Lambda^2}^{s_0} \frac{f(s-i\varepsilon)}{s-z} ds \right)$$

- If $\lim_{\Lambda^2 \to +\infty} \frac{1}{2i\pi} \oint_{|s|=\Lambda^2} \frac{f(s)}{s-z} ds = 0$

$$\Rightarrow \quad f(z) = \frac{1}{2i\pi}\left(\int_{s_0}^{\infty} \frac{f(s+i\varepsilon) - f(s-i\varepsilon)}{s-z} ds \right)$$

- Unsubtracted dispersion relation: $f(s) = \frac{1}{\pi}\int_{s_0}^{\infty} \frac{\text{Im}\, f(s')}{s' - s - i\varepsilon} ds'$

- f can be reconstructed everywhere from the knowledge of Im f along the cut

2.2 Principles of a dispersion relation

- Subtractions : If convergence of f not fast enough

  ⇨ subtract in s = a (a < s_0)

DR for $\frac{f(s)-f(a)}{s-a}$ also analytic

$$\frac{f(s)-f(a)}{s-a}=\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\mathrm{Im}\left(\frac{f(s')-f(a)}{s'-a}\right)}{s'-s-i\varepsilon}ds'$$

⇨ $$f(s)=f(a)+\frac{s-a}{\pi}\int_{s_0}^{\infty}\frac{\mathrm{Im}\, f(s')}{(s'-a)(s'-s-i\varepsilon)}ds'$$

Once subtracted dispersion relation

- Dispersion relation with n subtractions

$$f(s)=P_{n-1}(s)+\frac{(s-\bar{s})^n}{\pi}\int_{s_0}^{\infty}\frac{\mathrm{Im}\, f(s')}{(s'-\bar{s})^n(s'-s-i\varepsilon)}ds'$$