

Most of the work has been done with A. Grau and Y.N. Srivastava. Applications to the total cross-section energy behaviour was done also with R.M. Godbole and recently with O. Shekhovtsova.

Our aim is to understand the role played by soft gluon resummation in the infrared region at very high energy. From the very beginning I have been convinced of the importance of the Bloch-Nordsieck (BN) result of infinite number of soft photons emitted in any reaction between charged particles [?]. In QED the problem is a classical one, which had been posed by BN in 1937, solved by Schwinger in 1948 [?] from the point of view of the finiteness of the result, and dealt with to all orders in  $\alpha_{QED}$  during the fifties and early sixties. Soft Gluon k(t)-Resummation and the Froissart bound.

Having worked with Touschek on infrared radiative corrections in QED, I became convinced of the importance of the infrared region also in strong interactions and through the years have refined our understanding of the role played by resummation in the infrared region. This role is obviously recognized by many. How to implement it is less obvious. Presently we are investigating this role in the calculation of total cross-sections where the large distance scattering plays a dominant role. Not all is solved in our model, but we believe that the role played by the infrared limit in complying with the Froissart bound is clear [?]. The description of how this is accomplished will be described in this seminar.

**The Froissart bound** The origin of the Froissart bound is intimately related to confinement, namely to the fact the confining potential of strong interactions which is responsible for the existence of hadronic bound states imposes a finite limit to the region of interaction. All the derivations of the Froissart bound rely on an exponential or better gaussian effective cut-off in space. Such effective cut-off is reflected in the existence of a maximum angular momentum value,  $L_{max}$ , in the partial wave expansion of the scattering amplitude. The existence of such a maximum angular momentum together with an energy dependence of the partial wave amplitudes, which should be at maximum a polynomial in the invariant squared center of mass energy  $s$ , leads to the Froissart bound, namely to the fact that the total hadronic cross-section cannot increase more than a squared logarithm of the energy, i.e.

$$\sigma_{tot} \leq (\ln s)^2 \tag{1}$$

The Froissart bound reflects the concept of a finite region of interaction between hadrons, region, which at high energy, is represented in the transverse plane by a maximum impact parameter value, beyond which the colliding hadrons do not see each other any more. In QCD language, this means that no gluons are exchanged. If one can define such a  $b_{max}$  then through a geometrical argument, one writes the cross-section as

$$\sigma_{total} \sim \pi b_{max}^2 \quad (2)$$

and then the problem is to obtain the high energy limit of  $b_{max}$ . Analyticity considerations and the asymptotic limit of the Legendre functions of the partial wave expansion, lead to the fact that  $L_{max} \sim \ln a_{Lmax}$  and since the partial wave amplitudes can increase at most like a power, one immediately obtains  $b_{max} \sim L_{max}/k \sim \ln s/k$ , where  $k$  is the c.m. momentum. Basically the power law rise of the partial wave amplitudes is transformed into a logarithmic limiting behaviour and then, because of the existence of a maximum angular momentum value, into an impact parameter limitation. The geometrical argument then easily puts this into a limit on the scattering cross-section.

It should be noticed that before Froissart, and thus before Froissart, Martin and Lukakzu [?, ?] obtained the limit of Eq. ??, Heisenberg in 1952 had already used the geometrical argument to calculate the total cross-section of emission of pions as

$$\sigma_{tot} = \frac{\pi}{m_\pi^2} (\ln s)^2 \quad (3)$$

I shall reproduce Heisenberg's argument in the following paragraph, here I continue on a general discussion of the Froissart bound in QCD.

What I mentioned about FL shows why attempts to obtain the asymptotic behaviour of the total cross-section from perturbative QCD are bound to fail unless the very large distance behaviour, which is related to the confinement problem, is dealt with. What is the perturbative tool at the disposal of a QCD worker? We believe it is resummation. However, the problem with resummation, as it has been and is still commonly used, lies in the infrared region. Our opinion is that such tool is soft gluon resummation, sometimes called the Sudakov form factor, a rather inadequate definition according to us. The reason why is that unless the infrared region is accessed, the suppression coming from the Sudakov factor is not sufficient to produce the cut-off in  $b$ -space which would signal confinement. Indeed the Sudakov form factor is

always used with a lower momentum cut-off and can only give rise to double logarithmic terms. Whence exponentiated, as we shall show, these terms give a power law suppression (with anomalous dimensions) and NOT a gaussian or exponential cut-off as results from the Froissart bound.

On the other hand, the suppression a' la Bloch and Nordsieck is generated by the  $k \rightarrow 0$  limit. It thus should be clear that large distances require the treatment of very small momenta.

We shall now revisit soft resummation and identify where our point of view differs from the usual treatment. We shall start from the probability that in a given reaction between charged or colored (in the case of QCD) particles emit soft quanta, let them be photons or gluons. Assuming that one can neglect the recoil of the emitting particles, the emitted quanta can be described by a Poisson distribution, as in independent emission. This is what was shown by Bloch and Nordsieck (BN). BN were interested in the distribution of the overall energy emitted in the scattering. This distribution, the energy distribution is easily obtained from the probability of radiation with overall energy-momentum  $K_\mu$ . This distribution is given by

$$d^4P(K) = \frac{d^4K}{(2\pi)^4} \int d^4x \exp[-iK \cdot x - h(x)] \quad (4)$$

with

$$h(x) = \int d^3\bar{n}(k)[1 - e^{ik \cdot x}] \quad (5)$$

A result like the above, can be obtained semi-classically, using the methods of statistical mechanics. To proceed further one needs to know the single quanta distribution, for QED this is given as

$$d\bar{n}(k) = \frac{\alpha}{2\pi^2} \frac{d^3k}{2k} \sum_{i,j} \frac{\epsilon_i \epsilon_j p_{i,\mu} p_j^\mu}{(p_i \cdot k)(p_j \cdot k)} \quad (6)$$

Integrating over the spatial dimensions, and applying analyticity considerations a closed form is obtained as

$$\frac{dP(\omega)}{\omega} = \int \frac{d\omega}{\omega} \exp[i\omega t - h(t, E)] = \mathcal{N} \left(\frac{\omega}{E}\right)^\beta(E) \quad (7)$$

where  $\mathcal{N}$  is a normalization factor and  $\beta(E)$  is defined so that

$$\int_\Omega d^3\bar{n}(k) = \beta(E) \frac{dk}{k} \quad (8)$$

namely the radiator factor  $\beta(E)$  is the integral of the single quantum, photon in QED, distribution over the photon directions. In QED, for scattering of two charged particles into a neutral final state, if  $E$  is the electron energy in the c.m. system one has

$$\beta(E) = \frac{\alpha_{QED}}{\text{some } \pi} \ln \frac{2E}{m_e} \quad (9)$$

Before going to the transverse momentum distribution, notice the appearance of the scale  $E$ . Since the dependence is logarithmic, its exact value is not so important, its physical meaning being the maximum energy allowed for single photon emission, thus it cannot exceed the energy of the emitting particle.

One thus see the appearance of the large logarithms which can easily be resummed in QED. Integrating the expression for  $d^4P$  in the energy and longitudinal momentum variables, one obtains the probability distribution in the transverse momentum variable,

$$d^2P(K_\perp) = \frac{d^2K_\perp}{(2\pi)^2} \int d^2b \exp[-iK_\perp \cdot b - h(b)] \quad (10)$$

with

$$h(b) = \int_0^E dk_t \frac{\alpha}{k_t} \ln \frac{2E}{k_t} [1 - e^{ik_t \cdot b}] \quad (11)$$

Unlike the case of the energy variable, the transverse momentum distribution cannot be obtained in closed form. For QED processes, the spatial momentum distribution can be well approximated by the first order distribution, but when the coupling is large, this is not true anymore. Thus, in 1976 we proposed an approximation for this distribution [?] which had a gaussian fall off at highvalues of the transverse momentum. In our approximation , the coupling of the emitted quanta to the emitting currents was taken to be large, but not energy dependent. We could thus exploit, for the derivation of the gaussian cut-off, the infrared region without worrying about the behaviour of the coupling in this region.

Shortly after our proposal, Dokshitzer, Troyan and D'yakonov proposed in QCD the same expression as given by Eq. ?? but with a running coupling constant, and so did Parisi and Petronzio , and Soper and Collins, with different emphasis on different processes. However, no evidence of a cut-off appeared in these approaches, since the infrared region was not accessed by the integrals. The function was

$$h(b, \mu) = \int_\mu^E \frac{d^2k_t}{k_t} \alpha_s(k_t^2) \ln \frac{2E}{k_t} \quad (12)$$

which is an approximation to the actual expression one obtains from Eq. ???. The approximation consists in neglecting the second term in the square bracket, which is quite acceptable if the range of integration does not include very small values of the momentum  $k_t$  or or, as it is said in [?] if there are no singularities in the infrared region. But something singular is happening in the infrared region, namely confinement, and by using this approximation one is led to a loss of information on the infrared region, and, in our mind, one renders resummation nothing more than a useful but uninteresting correction. Recently, this point of view has been changing. Attention to the infrared region has surfaced. Our approach to resummation instead focuses on the infrared region, and how to exploit it to study long distance effects. To do so, one needs to retain the full integration region and of course also the two terms in the square bracket which cancel the infrared divergence coming from the gluon propagator. The two terms in the square bracket do not cancel whatever divergence the coupling might have in the infrared.

To deal with the integral in its entirety, including the zero momentum limit, we have made a toy model to describe the coupling of very soft gluons to the quark current. We expect that when the soft gluon momentum is larger than  $\Lambda_{QCD}$  one can safely use the AF expression for the coupling, but for  $k_t \leq \Lambda_{QCD}$  we propose to use

$$\alpha_{IR}(k_t) \sim k_t^{-2p} \quad (13)$$

This expression for the coupling in the infrared region allows to extend the integration down to zero, namely to use the original expression

$$h(b, E) = \int_0^E \frac{d^2 k_t}{k_t} \alpha_s(k_t^2) \ln \frac{2E}{k_t} [1 - J_0(k_t b)] \quad (14)$$

provided  $p < 1$ . The choice of an expression like the one given in Eq. ??? follows the requirement of integrability for the soft gluon spectrum and, at the same time, the presence of a singularity, as we can expect to exist in the infrared. In terms of a one-gluon exchange potential, such coupling can be interpreted as the one which would give a one gluon exchange potential rising at large distances as  $V(r) \sim r^{2p-1}$ , which, for  $p < 1$  is a confining potential. Thus, albeit in a far from rigorous manner, we have established a correspondence between a confining potential and the singular infrared behaviour in soft gluon resummation. Further understanding of such connection, comes from studying in detail how the function  $h(b, E)$  depends on the variable  $b$ , the conjugate of the soft gluon transverse momentum.

The proposed expression for the infrared behaviour of the coupling shows immediately its value, if we study the part of the integral which was indeed neglected by DDT and Parisi Petronzio, namely

$$h(b, E) = \int_0^\mu \frac{d^2 k_t}{k_t} \alpha_s(k_t^2) \ln \frac{2E}{k_t} [1 - J_0(k_t b)] + \Delta(\mu, E) \approx b^2 \int_0^\mu k_t dk_t \alpha_s(k_t^2) \ln \frac{2E}{k_t} + \Delta(\mu, E) \quad (15)$$

with

$$\Delta(\mu, E) = \int_\mu^E \frac{d^2 k_t}{k_t} \alpha_s(k_t^2) \ln \frac{2E}{k_t} \quad (16)$$

is the usual Sudakov form factor for the transverse momentum. Neglecting the  $J_0$  in the argument of the  $\Delta$  integral, allows to perform the integral in closed form, so that

$$\Delta(\mu, E) = \text{expression with the double logarithms} \quad (17)$$

In Eq. ??, we have expanded the  $J_0$  for small values of  $k_t$ , which correspond to large  $b$ -values and retained only the  $b^2 k_t^2$  terms. This is an approximation, but it shows the appearance of the cutoff in  $b$ -space which will be necessary to obtain a behaviour of the total cross-section complying with the Froissart bound. In this region, infact, we shall then have that  $\exp[-h(b, \mu)] \sim \exp[-(b/b_0)^2]$ .

One can study the function  $h(b, E)$  for a generic value of the scale  $E$ . To include in our study also the perturbative region, where indeed the integral is usually evaluated, we approximate the coupling as follows:

$$\alpha_{effective} = \frac{bp}{\ln[1 + (\frac{k_t^2}{\Lambda_{QCD}^2})^p]} \quad (18)$$

with  $b = 12\pi/(11N_f - 2N_c)$ . Then for  $k_t \gg \Lambda_{QCD}$  one has  $\alpha_{effective} \simeq \alpha_s^{AF}$ , whereas for  $k_t \ll \Lambda_{QCD}$

$$\alpha_{IR} = bp(\frac{\Lambda_{QCD}^2}{k_t^2})^p \quad (19)$$

With such an expression for instance, we can calculate the integral often discussed by Dokshitzer, namely

$$\alpha_0 = \frac{\int_0^\mu d^2 k_t \alpha_s(k_t^2)}{\int_0^\mu d^2 k_t} = \int_0^\mu \alpha_{IR}(k_t^2) \frac{d^2 k_t}{\mu^2} \propto \frac{1}{1-p} (\frac{\mu^2}{\Lambda_{QCD}^2})^{1-p} \quad (20)$$

For  $\Lambda_{QCD} = \mu$ , we get  $\alpha_0 = bp/(1 - p) \approx 1/(1 - p)$ . The phenomenology of total cross-sections which we have applied our model to, indicates values of  $p \approx 3/4$ , hence  $\alpha_0 = bp4/3 \approx 4/3$ , since  $bp \approx 1$  phenomenologically. However as  $p$  get close and closer to 1, we see that  $\alpha_0$  becomes larger and larger, reflecting the presence of a singularity at  $p = 1$ .

Our ansatz for the infrared behaviour gives a  $b$ -dependence different from other often proposed expressions, like the commonly used *frozen*  $\alpha_s$  expression, which goes to a constant in the infrared limit. As discussed in various papers, and to begin with in 1999, the frozen  $\alpha_s$  expression does not introduce anything new, even if the integral is extended down to zero, namely  $h(b, E)$  is still only a function of  $b$  through a logarithmic dependence, unlike the singular case where  $h(b, E)$  becomes proportional to  $b^2$  as  $b$  becomes large. We shall reoproduce in one slide the approximate expressions we have obtained and which are very close to their numerical evaluations.

Having discussed our revisitation of soft gluon resummation, we now go to describe its impact on the total cross-section calculations.

**Physical consequences of the IR soft gluon emission** As for Initial State Radiation in QED, soft gluon emission from the initial state generates an momentum imbalance in the final state . In  $e^+e^-$ , the emission of one or more photons produces an acollinearity, and the same effect can be expected in parton parton collisions. This acollinearity can produce different effects, like a diminished probability of collision when the parton beam from one proton meets the other parton beam from the other proton, and may also produce the recently observed asymmetric effects observed in heavy ion collisions, as we describe qualitatively in another paragraph. The acollinearity of the scattering partons induced by soft gluon emission is what trasforms the mini-jet contribution to the toal cross-section into a behaviour consistet withthe Froissart bound. We have described how the probability for soft gluon emission in b-spacemay exhibits a gaussian or exponential cut-off. A maximum radius of interaction,  $b_{max}$  then arises and when included in an eikonal representation for the cross-section, the effect is the same as the one described in the case of the Froissart bound, the high energy behaviour of the cross-section is related to the maximum istance at which collisions are still possible, through a logarithm. Power law behaviour of the amplitudes or in the case at hand of the parton parton cross-sections are transformed into a logarithmic behaviour if there is a b-space cut-off. We shall first describe this

mechanism for the eikonal minijet model with soft gluon resummation which we have developed. After showing the mechanism which induces saturation, we shall describe in detail the phenomenological applications.

Eikonal models start from the b-space representation of the scattering amplitude

$$F(s, t) = \int d^2b f(b, s) = \int d^2b e^{iq \cdot b} [1 - e^{i\chi(b, s)}] \quad (21)$$

and thus to obtain

$$\frac{d^2\sigma_{elastic}}{d^2b} = |1 - e^{i\chi(b, s)}|^2 \quad (22)$$

and through the optical theorem to

$$\sigma_{total} = 2 \int^2 b \Re[1 - e^{i\chi(b, s)}] = 2 \int d^2b [1 - \Re e\chi(b, s) e^{-\Im m\chi(b, s)}] \quad (23)$$

Using the two above equations, one immediately obtain the total inelastic cross-section, i.e.

$$\sigma_{inel} = \int d^2b [1 - e^{-\Im m\chi(b, s)}] \quad (24)$$

The above equation gives a chance to calculate the total cross-section through models for the imaginary part of the eikonal function  $\chi(b, s)$ . It is fortunate that with very simple assumptions, the imaginary part can be related to the average number of inelastic collisions at impact parametr b and given c.m. energy s. Consider in fact all the partion-parton collisions which can take place during the scattering, with different distributions in  $n_1$  collisions,  $n_2$  collisions etc. Let these collisions be all independent from one another, one can postulate that they follow a Poisson distribution around an average number of collisions  $\bar{n}(b, s)$

$$P(n, \bar{n}) = \frac{\bar{n}^n}{n!} e^{-\bar{n}} \quad (25)$$

Summing over the number of collisions from  $n = 1$ , one obtains the total probability for inelastic collisions

$$\sigma_{inelastic}(s) = \int d^2b [1 - e^{-\bar{n}(b, s)}] \quad (26)$$

This equation allows the identification  $\bar{n}(b, s) = \Im m\chi(b, s)$  and building a model for the average number of inelastic collisions, one is then one step ahead for having a model for the total cross-section. In our work, so far, we

have neglected the real part of the eikonal. This is in part consistent with the phenomenological result that the ratio of the real to the imaginary part of the elastic scattering amplitude is very small. However, neglecting the real part at every single value of  $b$  does introduce an error. This approximation is probably at the root of the well known problem of the eikonal mini-jet models, in which if one fits the total cross-section by fixing the given model parameters, then the elastic cross-section calculated with the same set of parameters, becomes too large. Alternatively one could fix the parameters fitting the elastic cross-section, but then with the same parameters the total cross-section is too small.

In our present application of the Eikonal Mini jet Model (EMM), we ignore this problem, and concentrate on obtaining a good description and understanding of the energy behaviour of the total cross-section, assuming the Real part of the eikonal function to be zero everywhere in  $b$ -space. I repeat that this is not true and that this approximation is probably at the origin of the overestimate of the elastic cross-section which comes from minijet models. If we do this, then once we calculate  $\bar{n}(b, s)$ , we can calculate the total cross-section.

Following the usual pattern, the average number of collision can be factorized in an impact parameter distribution and a scattering cross-section. At the parton level, we can write

$$\bar{n}(b, s) = \int dx_1 dx_2 \int d^2 k_t f_{i,A}(x_1, Q^2, k_t) j_{j,B}(x_2, Q^2, k_t) \int d^2 p_t \frac{d^2 \hat{\sigma}}{d^2 p_t} \quad (27)$$

where we have assumed that partons of momentum  $x_i$  and relative transverse momentum  $k_t$  scatter with elementary point-like cross-section  $\frac{d^2 \hat{\sigma}}{d^2 p_t}$  into a final parton parton state with parton transverse momentum  $p_t$ .

## 1 Added material

### Heisenberg derivation for the total cross-section

**The average of  $\alpha_s$**  Our model for the strong coupling constant allows for calculation of the average value of  $\alpha_s$  defined as

$$\bar{\alpha}_s = \frac{\int_0^{q_{max}} k_t^2 dk_t \alpha_s(k_t^2)}{\langle k_t^2 \rangle} \quad (28)$$

**The scales** In our model we distinguish two different transverse momentum scales, one from partons which have undergone a hard scattering, and one from soft partons which do not see the other partons, and are only soft gluons emitted during the hard scattering. To distinguish them, we define call  $k_t$  the soft gluon momentum and  $p_t$  the perturbative parton momentum. This is a simplification, but it is useful when wanting to use the tools at our disposal, LO or NLO QCD and soft resummation. Clearly the distinction between these two type of partons is not something as clear cut as we represent it, but it is a toy model, whose usefulness as we shall see is justified a posteriori. Let us now see how we deal with these two different regimes:

- $p_{tmin}$ : a perturbative scale such that for  $p_t \geq p_{tmin} \simeq 1 - 2 \text{ GeV}$  all hadrons appearing in the final state with such a transverse momentum can be treated perturbatively, and are generally called a mini-jet, namely a jet with a small, yet still perturbative, transverse momentum; the production cross-section for such mini-jets can be calculated from the  $2 \rightarrow 2$  perturbative parton-parton cross-section with  $\alpha_s(p_t^2)$  evaluated using the usual asymptotic freedom (AF) expression for QCD. Clearly using the AF expression can be a bit hazardous at  $p_t \simeq 1 \text{ GeV}$ , but it can still be considered acceptable; the parton-parton cross-section integrated over all possible initial momenta of the colliding collinear partons gives rise to the so called mini-jet cross-section.
- $q_{max}(x_1, x_2, p_t)$ : a scale which represents the maximum transverse momentum which can be carried away by a single soft gluon emitted during the collision between the two partons of longitudinal momenta  $x_1$  and  $x_2$  of the soft gluon momentum scale,  $k_t$  which requires a special treatment, depending on the energy scale of the process. This scale arises because LO parton-parton process are accompanied by soft gluon emission. The upper limit in the single soft gluon integral is usually appearing only through a logarithm and its exact value is not very important, however it should be of the order of magnitude of the energy of the colliding particles. One can have a good estimate, by using the single gluon kinematic limit, as it was done in the early times, namely

$$q_{max}(x_1, x_2) = \frac{\hat{s}}{2} \left(1 - \frac{Q^2}{\hat{s}}\right) \frac{1}{\sqrt{1 + z \sinh^2 \eta}} \quad (29)$$

with  $\hat{s} = sx_1x_2$ ,  $Q^2$  is the squared invariant mass of the outgoing pair

with partons, each of them with transverse momentum  $p_t$ , in the no-recoil approximation,  $z = Q^2/\hat{s}$ ,  $\eta$  is the pseudorapidity of the outgoing partons. Typically one has that  $q_{max} \sim p_t$ , since this is the scale of the parton-parton process, therefore  $q_{max}$  could still be a perturbative scale.

- $\Lambda_{QCD}$ : below this value, is the region crucial for the asymptotic limit of the total cross-section, according to the model we have developed. In this region the AF limit of  $\alpha_s$  needs to be modified as to at least mimic the confinement process, our ansatz is to use

$$\alpha_{IR} = c \left( \frac{\Lambda_{QCD}^2}{k_t^2} \right)^p \quad (30)$$

namely an expression which is singular but integrable. Such an expression is to be used when the distances involved in the scattering processes are such as to require integration in this region, we shall discuss this in detail in the part dedicated explicitly to resummation.

The interplay of these scales in a given physical process will be illustrated when dealing with our model for the total cross-section.

**Outlook** The list of topics to be discussed in the future is the following:

- application of our model to nuclear or heavy ion physics Two major issues are relevant from the point of view of our model and interaction of protons with nuclei, namely
  1. Cosmic rays physics: the extension of the model to  $p - air$  interactions and the inclusion of our model prediction for total cross-section for proton, pion and photon interactions. We shall illustrate the present situation of data and our model prediction for all these processes in Fig. 1.
  2. the calculation of  $p - air$  cross-sections using our model for proton interactions and a new ansatz for the nuclear distribution as seen by an incoming proton of ultra high energy,  $E_{lab} \sim 10^{19} eV$  and larger. The question here is how extremely high energy protons see the nuclei with which they interact. A lesson in this sense comes from the recent observation at LHC in  $P_b - P_b$  collisions of one high energy jet balanced by a diffuse distribution of low  $p_t$

events. This is interpreted as the fact that one of the quarks from the scattering is emitted as such pseudorapidity as not to interact with the heavy medium, while the other is absorbed and loses all its energy. There must also be some initial state radiation which gives an imbalance such as to produce such final state asymmetry. If the initial state is not collinear, because the quarks radiate strongly, this will prevent a back-to-back pair jets in the final state. Now, if one of the quarks is emitted so as to escape the dense nuclear matter, the same may not be true for the other quark which will dissipate all its energy within the medium. The final energy momentum balance will be insured by initial state radiation as well as jet dissipation.

- Heavy Ion collisions at very high energy and the occurrence of QGP: we take the point of view that when very high energy nuclear matter interacts, through heavy ion collisions or ultra high energy cosmic rays, what happens is that nucleons from the interacting nuclei deeply penetrate each other and quarks from one proton in one nucleus see quarks in another nucleus. When this happens, soft gluons are emitted from the initial scattering quarks and the final state reflects such soft gluon abundant emission. The infrared limit of soft  $k_t$ -gluon resummation plays a role. QGP from this point of view is that quarks from different nuclei interact.

## References

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