Massive gravitons in arbitrary spacetimes

Mikhail S. Volkov

LMPT, University of Tours, FRANCE

Orsay, *Modern aspects of Gravity and Cosmology,*
8-th November 2017

C.Mazuet and M.S.V. arXiv: 1708.03554

- Linear massive gravitons in curved space
- Cosmology
- Black holes

Massive fields in curved space

Spin 0. One has in Minkowski space

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = M^2\Phi$$

To pass to curved space one replaces

$$\eta_{\mu\nu} \Rightarrow g_{\mu\nu}, \qquad \partial_\mu \Rightarrow \nabla_\mu$$

which gives

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = M^2\Phi$$

Similarly for spins 1/2 (Dirac), 1 (Proca), 3/2 (Rarita-Schwinger).
The procedure fails for the massive spin 2.

Massive spin 2 in flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \Rightarrow \qquad E_{\mu\nu} \equiv 2\delta G_{\mu\nu} + M^2(h_{\mu\nu} - \lambda h\,\eta_{\mu\nu}) = 0,$$

or explicitly

$$\begin{aligned} E_{\mu\nu} &\equiv \partial^\sigma\partial_\mu h_{\sigma\nu} + \partial^\sigma\partial_\nu h_{\sigma\mu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h \\ &+ \eta_{\mu\nu}(\Box h - \partial^\alpha\partial^\beta h_{\alpha\beta}) + M^2(h_{\mu\nu} - \lambda h\,\eta_{\mu\nu}) = 0. \end{aligned}$$

One has $\partial^\mu\delta G_{\mu\nu} \equiv 0 \Rightarrow 4$ vector constraints

$$\mathcal{C}_\nu \equiv \partial^\mu E_{\mu\nu} = M^2(\partial^\mu h_{\mu\nu} - \lambda\partial_\nu h) = 0,$$

and also

$$\mathcal{C}_5 = \partial^\nu\mathcal{C}_\nu + M^2\eta^{\mu\nu}E_{\mu\nu} = M^2(1-\lambda)\Box h + M^4(1-4\lambda)\,h = 0,$$

which becomes constraint if $\lambda = 1$.

Fierz-Pauli: $\lambda = 1$

$$
\begin{aligned}
(\Box + M^2) h_{\mu\nu} &= 0, \\
\partial^\mu h_{\mu\nu} &= 0, \\
h &= 0,
\end{aligned}
$$

$\Rightarrow 10 - 5 = 5$ propagating DoF. For $\lambda \neq 1$ there are 6 DoF.
Passing to curved space via $\eta_{\mu\nu} \to g_{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$ yields

$$
\begin{aligned}
E_{\mu\nu} &\equiv \nabla^\sigma \nabla_\mu h_{\sigma\nu} + \nabla^\sigma \nabla_\nu h_{\sigma\mu} - \Box h_{\mu\nu} - \nabla_\mu \nabla_\nu h \\
&+ g_{\mu\nu}(\Box h - \nabla^\alpha \nabla^\beta h_{\alpha\beta}) + M^2 (h_{\mu\nu} - h\, g_{\mu\nu}) = 0.
\end{aligned}
$$

This implies the 5 constraints

$$
\begin{aligned}
&\mathcal{C}_\nu \equiv \nabla^\mu E_{\mu\nu} = M^2 (\nabla^\mu h_{\mu\nu} - \nabla_\nu h) = 0, \\
&\mathcal{C}_5 = \nabla^\nu \mathcal{C}_\nu + M^2 g^{\mu\nu} E_{\mu\nu} = -3M^4\, h = 0
\end{aligned}
$$

but ONLY in vacuum or in Einstein space, for $R_{\mu\nu} = \Lambda g_{\mu\nu}$.
Otherwise 6 DoF. What to do ?

Linear theory from the nonlinear one

Ghost-free massive gravity

Let $g_{\mu\nu}$ and $f_{\mu\nu}$ be the physical and reference metrics and

$$\gamma^{\mu}{}_{\sigma}\gamma^{\sigma}{}_{\nu} = g^{\mu\sigma} f_{\sigma\nu}, \qquad \gamma_{\mu\nu} = g_{\mu\sigma}\gamma^{\sigma}{}_{\nu}, \qquad [\gamma] = \gamma^{\sigma}{}_{\sigma}.$$

The equations are /dRGT, 2010/

$$\begin{aligned} \mathbf{E}_{\mu\nu} &\equiv G_{\mu\nu}(g) + \beta_0\, g_{\mu\nu} + \beta_1([\gamma]\, g_{\mu\nu} - \gamma_{\mu\nu}) \\ &+ \beta_2\, |\gamma|\, ([\gamma^{-1}]\, \gamma^{-1}_{\mu\nu} - \gamma^{-2}_{\mu\nu}) + \beta_3\, |\gamma|\, \gamma^{-1}_{\mu\nu} = 0. \end{aligned}$$

Perturbing $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ then $\mathbf{E}_{\mu\nu} \to \mathbf{E}_{\mu\nu} + \delta\mathbf{E}_{\mu\nu}$ with

$$\delta\mathbf{E}_{\mu\nu} = \delta G_{\mu\nu} + \beta_0\, \delta g_{\mu\nu} + \beta_1([\delta\gamma]\, g_{\mu\nu} + [\gamma]\, \delta g_{\mu\nu} - \delta\gamma_{\mu\nu}) + \ldots$$

where $\quad \delta\gamma^{\mu}{}_{\sigma}\gamma^{\sigma}{}_{\nu} + \gamma^{\mu}{}_{\sigma}\delta\gamma^{\sigma}{}_{\nu} = \delta g^{\mu\sigma} f_{\sigma\nu} \quad \Leftrightarrow \quad \delta\gamma\gamma + \gamma\delta\gamma = \delta g^{-1} f.$

Solution for $\delta\gamma^{\mu}{}_{\sigma}$ in terms of $\delta g^{\mu\sigma}$ is <u>very complicated</u> /Deffayet et al./

Ghost-free massive gravity in tetrad formalism

Introducing two tetrads ${e^a}_\mu$ and ${\phi^a}_\mu$ such that

$$g_{\mu\nu} = \eta_{ab}{e^a}_\mu {e^b}_\nu, \qquad f_{\mu\nu} = \eta_{ab}{\phi^a}_\mu {\phi^b}_\nu,$$

one has

$$\gamma^a{}_b = {\phi^a}_\sigma {e_b}^\sigma, \qquad \gamma_{ab} = \eta_{ac}\gamma^c{}_b = \gamma_{ba}$$

and the equations

$$\begin{aligned} \mathbf{E}_{ab} &\equiv G_{ab} + \beta_0\,\eta_{ab} + \beta_1([\gamma]\,\eta_{ab} - \gamma_{ab}) \\ &+ \beta_2\,|\gamma|\left([\gamma^{-1}]\,\gamma^{-1}_{ab} - \gamma^{-2}_{ab}\right) + \beta_3\,|\gamma|\,\gamma^{-1}_{ab} = 0. \end{aligned}$$

The idea is to linearize with respect to tetrad perturbations

$${e^a}_\mu \to {e^a}_\mu + \delta{e^a}_\mu \quad \text{with} \qquad \delta{e^a}_\mu = X^a{}_b {e^b}_\mu$$

and then project to ${e_a}^\mu$ and express everything in terms of

$$\boxed{X_{\mu\nu} = \eta_{ab}{e^a}_\mu \delta{e^b}_\nu} \qquad \Rightarrow \qquad \delta g_{\mu\nu} = X_{\mu\nu} + X_{\nu\mu}$$

Equations in the generic case

$$E_{\mu\nu} \equiv \Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0$$

with the kinetic term

$$\begin{aligned}
\Delta_{\mu\nu} &= \frac{1}{2}\nabla^\sigma\nabla_\mu(X_{\sigma\nu} + X_{\nu\sigma}) + \frac{1}{2}\nabla^\sigma\nabla_\nu(X_{\sigma\mu} + X_{\mu\sigma}) \\
&- \frac{1}{2}\Box(X_{\mu\nu} + X_{\nu\mu}) - \nabla_\mu\nabla_\nu X \\
&+ g_{\mu\nu}\left(\Box X - \nabla^\alpha\nabla^\beta X_{\alpha\beta} + R^{\alpha\beta}X_{\alpha\beta}\right) \\
&- R^\sigma_\mu X_{\sigma\nu} - R^\sigma_\nu X_{\sigma\mu}
\end{aligned}$$

and the mass term

$$\begin{aligned}
\mathcal{M}_{\mu\nu} &= \beta_1\left(\gamma^\sigma{}_\mu X_{\sigma\nu} - g_{\mu\nu}\,\gamma^{\alpha\beta}\,X_{\alpha\beta}\right) \\
&+ \beta_2\,\{-\gamma^\alpha{}_\mu\gamma^\beta{}_\nu\,X_{\alpha\beta} - (\gamma^2)^\alpha{}_\mu\,X_{\alpha\nu} + \gamma_{\mu\nu}\,\gamma_{\alpha\beta}\,X^{\alpha\beta} \\
&\qquad +[\gamma]\,\gamma^\alpha{}_\beta\,X_{\alpha\nu} + ((\gamma^2)_{\alpha\beta}\,X^{\alpha\beta} - [\gamma]\,\gamma_{\alpha\beta}\,X^{\alpha\beta})\,g_{\mu\nu}\} \\
&+ \beta_3\,|\gamma|\,\left(X_{\mu\sigma}(\gamma^{-1})^\sigma{}_\nu - [X](\gamma^{-1})_{\mu\nu}\right)
\end{aligned}$$

$\gamma_{\mu\nu}$ is algebraically expressed in terms of background $G_{\mu\nu}$ and $g_{\mu\nu}$.

Equations for $\gamma_{\mu\nu}$

Background equations

$$G_{\mu\nu} + \beta_0\, g_{\mu\nu} + \beta_1([\gamma]\, g_{\mu\nu} - \gamma_{\mu\nu})$$
$$+\beta_2\, |\gamma|\, \big([\gamma^{-1}]\, \gamma^{-1}_{\mu\nu} - \gamma^{-2}_{\mu\nu}\big) + \beta_3\, |\gamma|\, \gamma^{-1}_{\mu\nu} = 0$$

can be viewed as cubic algebraic equations for $\gamma_{\mu\nu}$. For any $g_{\mu\nu}$ the solution is

$$\gamma_{\mu\nu}(g) = \sum_{n,k=0}^{\infty} b_{nk}(\beta_A)\, R^n\, (R^k)_{\mu\nu}\,,$$

There are special values of β_A for which the sum is finite.

How many propagating DoF are there ?

Constraints

There are 16 equations

$$E_{\mu\nu} \equiv \Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0$$

for 16 components of $X_{\mu\nu}$. The imply the following 11 conditions:

$$\Delta_{[\mu\nu]} = 0 \quad \Rightarrow \quad \mathcal{M}_{[\mu\nu]} = 0 \quad \Rightarrow \quad \text{6 algebraic constraints}$$

$$\mathcal{C}_\nu = \nabla^\mu E_{\mu\nu} = 0 \quad \Rightarrow \quad \text{4 vector constraints}$$

$$\begin{aligned}
\mathcal{C}_5 &= \nabla_\mu((\gamma^{-1})^{\mu\nu}\mathcal{C}_\nu) + \frac{\beta_1}{2}\,E^\alpha_{\ \alpha} + \beta_2\gamma^{\mu\nu}E_{\mu\nu} \\
&+ \beta_3\,\frac{|\gamma|}{g^{00}}\Big((\gamma^{-1})^{0\alpha}(\gamma^{-1})^{0\beta} - (\gamma^{-1})^{00}(\gamma^{-1})^{\alpha\beta}\Big) \\
&\times \left(E_{\alpha\beta} - \frac{1}{2}\,g_{\alpha\beta}(E^\sigma_{\ \sigma} - \frac{1}{g^{00}}\,E^{00})\right)\Bigg) = 0 \ \Rightarrow \quad \text{scalar constraint}
\end{aligned}$$

The number of DoF is $16 - 6 - 4 - 1 = 5$.

Two special models

Models I and II

Background equations

$$G_{\mu\nu} + \beta_0\, g_{\mu\nu} + \beta_1([\gamma]\, g_{\mu\nu} - \gamma_{\mu\nu})$$
$$+\beta_2\, |\gamma| \left([\gamma^{-1}]\, \gamma^{-1}_{\mu\nu} - \gamma^{-2}_{\mu\nu}\right) + \beta_3\, |\gamma|\, \gamma^{-1}_{\mu\nu} = 0$$

are non-linear in $\gamma_{\mu\nu}$. There are two exceptional cases:
<u>Model I</u>: $\beta_2 = \beta_3 = 0$,

$$G_{\mu\nu} + \beta_0\, g_{\mu\nu} + \beta_1([\gamma]\, g_{\mu\nu} - \gamma_{\mu\nu}),$$

which can be resolved with respect to $\gamma_{\mu\nu}$;
<u>Model II</u>: $\beta_1 = \beta_2 = 0$,

$$G_{\mu\nu} + \beta_0\, g_{\mu\nu} + \beta_3\, |\gamma|\, \gamma^{-1}_{\mu\nu} = 0,$$

which can be resolved with respect to $|\gamma|\gamma^{-1}_{\mu\nu}$.

Equations for the two special models

$$E_{\mu\nu} \equiv \Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0$$

with the kinetic term

$$\begin{aligned}
\Delta_{\mu\nu} &= \frac{1}{2}\nabla^\sigma\nabla_\mu(X_{\sigma\nu} + X_{\nu\sigma}) + \frac{1}{2}\nabla^\sigma\nabla_\nu(X_{\sigma\mu} + X_{\mu\sigma}) \\
&- \frac{1}{2}\Box(X_{\mu\nu} + X_{\nu\mu}) - \nabla_\mu\nabla_\nu X \\
&+ g_{\mu\nu}\left(\Box X - \nabla^\alpha\nabla^\beta X_{\alpha\beta} + R^{\alpha\beta}X_{\alpha\beta}\right) \\
&- R^\sigma_\mu X_{\sigma\nu} - R^\sigma_\nu X_{\sigma\mu}
\end{aligned}$$

and the mass term (M being the FP mass)

$$\begin{aligned}
\text{model I:} \qquad \mathcal{M}_{\mu\nu} &= \gamma_{\mu\alpha}X^\alpha_{\ \nu} - g_{\mu\nu}\,\gamma_{\alpha\beta}X^{\alpha\beta}, \\
\gamma_{\mu\nu} &= R_{\mu\nu} + \left(M^2 - \frac{R}{6}\right)g_{\mu\nu}\,; \\
\text{model II:} \qquad \mathcal{M}_{\mu\nu} &= -X_\mu^{\ \alpha}\gamma_{\alpha\nu} + X\gamma_{\mu\nu}\,, \\
\gamma_{\mu\nu} &= R_{\mu\nu} - \left(M^2 + \frac{R}{2}\right)g_{\mu\nu}\,.
\end{aligned}$$

Action

$$I = \frac{1}{2}\int X^{\nu\mu} E_{\mu\nu}\sqrt{-g}\, d^4x \equiv \int L\,\sqrt{-g}\, d^4x$$

(order of indices !) with $L = L_{(2)} + L_{(0)}$ where

$$\begin{aligned} L_{(2)} = & - \frac{1}{4}\nabla^\sigma \mathcal{X}^{\mu\nu}\nabla_\mu \mathcal{X}_{\nu\sigma} + \frac{1}{8}\nabla^\alpha \mathcal{X}^{\mu\nu}\nabla_\alpha \mathcal{X}_{\mu\nu} \\ & + \frac{1}{4}\nabla^\alpha \mathcal{X}\nabla^\beta \mathcal{X}_{\alpha\beta} - \frac{1}{8}\nabla_\alpha \mathcal{X}\nabla^\alpha \mathcal{X} \end{aligned}$$

with $\mathcal{X}_{\mu\nu} = X_{\mu\nu} + X_{\nu\mu}$ and $\mathcal{X} = \mathcal{X}^\alpha_{\ \alpha}$. One has in model I

$$\begin{aligned} L_{(0)} = & - \frac{1}{2} X^{\mu\nu} R^\sigma_{\ \mu} X_{\sigma\nu} \\ & + \frac{1}{2}(M^2 - \frac{R}{6})(X_{\mu\nu}X^{\nu\mu} - X^2) \end{aligned}$$

and in model II

$$\begin{aligned} L_{(0)} = & - \frac{1}{2} X^{\mu\nu} R^\sigma_{\ \mu} X_{\sigma\nu} - \frac{1}{2} X^{\mu\nu} R^\sigma_{\ \nu} X_{\sigma\mu} \\ & - \frac{1}{2} X^{\mu\nu} X_{\nu\alpha} R^\alpha_{\ \mu} + X R_{\mu\nu} X^{\mu\nu} + \frac{1}{2}(M^2 + \frac{R}{2})(X_{\mu\nu}X^{\nu\mu} - X^2) \end{aligned}$$

Constraints

Algebraic constraints

$$E_{\mu\nu} \equiv \Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0$$

are 16 equations for 16 components of $X_{\mu\nu}$. One has $\Delta_{\mu\nu} = \Delta_{\nu\mu}$ hence one should have

$$\mathcal{M}_{[\mu\nu]} = 0$$

which yields 6 algebraic conditions

$$\begin{array}{rrcl} \text{Model I:} & \gamma_{\mu\alpha} X^{\alpha}_{\ \nu} & = & \gamma_{\nu\alpha} X^{\alpha}_{\ \mu} \\ \text{Model II:} & X_{\mu}^{\ \alpha} \gamma_{\alpha\nu} & = & X_{\nu}^{\ \alpha} \gamma_{\alpha\mu} \end{array}$$

which reduce the number of independent components of $X_{\mu\nu}$ to 10.

Differential constraints, model I

with

$$\gamma_{\mu\nu} = R_{\mu\nu} + \left(M^2 - \frac{R}{6}\right) g_{\mu\nu}$$

one obtains the four vector constraints

$$\mathcal{C}^\rho \equiv (\gamma^{-1})^{\rho\nu} \nabla^\mu E_{\mu\nu} = \nabla_\sigma X^{\sigma\rho} - \nabla^\rho X + \mathcal{I}^\rho = 0$$

with

$$\mathcal{I}^\rho = (\gamma^{-1})^{\rho\nu} \left\{ X^{\alpha\beta}(\nabla_\alpha G_{\beta\nu} - \nabla_\nu \gamma_{\alpha\beta}) + \nabla^\mu \gamma_{\mu\alpha} X^\alpha_{\ \nu} \right\}$$

Next

$$\begin{aligned} \mathcal{C}_5 &\equiv \nabla_\rho \mathcal{C}^\rho + \frac{1}{2} E^\mu_{\ \mu} \\ &= -\frac{3}{2} M^2 X - \frac{1}{2} G^{\mu\nu} X_{\mu\nu} + \nabla_\rho \mathcal{I}^\rho = 0 \end{aligned}$$

$\Rightarrow$ the number of DoF is $10 - 5 = 5$.

Differential constraints, model II

With

$$\gamma_{\mu\nu} = R_{\mu\nu} - \left(M^2 + \frac{R}{2} \right) g_{\mu\nu}$$

one has

$$\mathcal{C}^\rho \equiv \gamma^{\rho\nu} \nabla^\mu E_{\mu\nu} = \Sigma^{\rho\nu\alpha\beta} \nabla_\nu X_{\alpha\beta} = 0$$

with $\Sigma^{\rho\nu\alpha\beta} \equiv \gamma^{\rho\nu}\gamma^{\alpha\beta} - \gamma^{\rho\beta}\gamma^{\nu\alpha}$ and

$$\begin{aligned} \mathcal{C}_5 &\equiv \nabla_\rho \mathcal{C}^\rho \\ &+ \frac{1}{2g^{00}} \Sigma^{00\alpha}{}_{\beta} \left(2E^{\beta}_{\ \alpha} - \delta^\beta_\alpha \left(E^{\sigma}_{\ \sigma} - \frac{1}{g^{00}} E^{00} \right) \right) = 0. \end{aligned}$$

This does not contain the <u>second time derivative</u> $\Rightarrow$ constraint.

Einstein space background

Einstein spaces, massless limit

If

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad \Rightarrow \quad \gamma_{\mu\nu} \propto g_{\mu\nu} \quad \Rightarrow \quad X_{\mu\nu} = X_{\nu\mu}$$

the equations reduce to

$$\Delta_{\mu\nu} + M_{\mathrm{H}}^2 (X_{\mu\nu} - X g_{\mu\nu}) = 0$$

where the Higuchi mass

$$\mathrm{I:} \qquad M_{\mathrm{H}}^2 = \Lambda/3 + M^2, \qquad \mathrm{II:} \qquad M_{\mathrm{H}}^2 = \Lambda + M^2.$$

If $M_{\mathrm{H}} = 0$ then equations admit the (diff) gauge symmetry

$$X_{\mu\nu} \to X_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)}$$

which reduces the number of DoF: $10 \to 2 =$ massless limit.

Einstein spaces, partially massless limit

For $M_{\rm H} \neq 0$ the divergence of the equations yields four constraints

$$\nabla^\mu X_{\mu\nu} = \nabla_\nu X,$$

equations reduce to

$$-\Box X_{\mu\nu} + \nabla_\mu \nabla_\nu X - 2R_{\mu\alpha\nu\beta} X^{\alpha\beta} + \Lambda X g_{\mu\nu} + M_{\rm H}^2 (X_{\mu\nu} - X g_{\mu\nu}) = 0.$$

Tracing this yields

$$(2\Lambda - 3M_{\rm H}^2) X = 0 \qquad \Rightarrow \qquad X = 0$$

$\Rightarrow 10 - 5 = 5$ DoF. If $M_{\rm H}^2 = 2\Lambda/3 \equiv M_{\rm PM}^2 \Rightarrow$ gauge symmetry

$$X_{\mu\nu} \to X_{\mu\nu} + \nabla_\mu \nabla_\nu \Omega + (\Lambda/3) g_{\mu\nu} \Omega$$

$\Rightarrow$ there remain only $4 = 10 - 1 - 1$ DoF $=$ partially massless case.

Short summary

- Six algebraic conditions and five differential constraints $\mathcal{C}^\rho = 0$ and $\mathcal{C}_5 = 0$ reduce the number of independent components of $X_{\mu\nu}$ from 16 to 5. This matches the number of polarizations of massive particles of spin 2.
- When restricted to Einstein spaces, the theory reproduces the standard description of massive gravitons.
- Unless in Einstein spaces, no massless (or partially massless) limit. For any value of the FP mass M the number of DoF on generic background is 5.

Cosmological background

FLRW cosmology

Line element

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\mathbf{x}^2$$

where $a(t)$ fulfills the Einstein equations

$$3\,\frac{\dot{a}^2}{a^2} = \frac{\boldsymbol{\rho}}{M_{\mathrm{Pl}}^2} \equiv \rho, \qquad 2\,\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\frac{\boldsymbol{p}}{M_{\mathrm{Pl}}^2} \equiv -p.$$

Here M_{Pl} is the Planck mass and $\boldsymbol{\rho}, \boldsymbol{p}$ are the energy density and pressure of the background matter.

Fourier decomposition

$$X_{\mu\nu}(t,\mathbf{x}) = a^2(t)\sum_{\mathbf{k}} X_{\mu\nu}(t,\mathbf{k})e^{i\mathbf{kx}}$$

where the Fourier amplitude splits into the sum of the tensor, vector, and scalar harmonics,

$$X_{\mu\nu}(t,\mathbf{k}) = X^{(2)}_{\mu\nu} + X^{(1)}_{\mu\nu} + X^{(0)}_{\mu\nu}$$

. The spatial part of the background Ricci tensor $R_{ik} \sim \delta_{ik}$ hence

$$X_{ik} = X_{ki}$$

$\Rightarrow X_{\mu\nu}$ has only 13 independent components. Assuming the spatial momentum $\mathbf{k}$ to be directed along the third axis, $\mathbf{k} = (0,0,\mathrm{k})$, the harmonics are

$$X^{(2)}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{D}_+ & \mathrm{D}_- & 0 \\ 0 & \mathrm{D}_- & -\mathrm{D}_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; X^{(1)}_{\mu\nu} = \begin{bmatrix} 0 & W^+_+ & W^+_- & 0 \\ W^-_+ & 0 & 0 & i\mathrm{kV}_+ \\ W^-_- & 0 & 0 & i\mathrm{kV}_- \\ 0 & i\mathrm{kV}_+ & i\mathrm{kV}_- & 0 \end{bmatrix},$$

$$X^{(0)}_{\mu\nu} = \begin{bmatrix} S^+_+ & 0 & 0 & i\mathrm{k}S^+_- \\ 0 & S^-_- & 0 & 0 \\ 0 & 0 & S^-_- & 0 \\ i\mathrm{k}S^-_+ & 0 & 0 & S^-_- - \mathrm{k}^2\mathrm{S} \end{bmatrix},$$

where $\mathrm{D}_\pm$, $\mathrm{V}_\pm$, S, $W^\pm_\pm$, $S^\pm_\pm$ are 13 functions of time. The equations split into three independent groups – one for the tensor modes $X^{(2)}_{\mu\nu}$, one for vector modes $X^{(1)}_{\mu\nu}$, and one for scalar modes $X^{(0)}_{\mu\nu}$.

Tensor sector

The effective action is

$$I_{(2)} = \int (K\dot{\mathrm{D}}_{\pm}^2 - U\mathrm{D}_{\pm}^2)\, a^3 dt$$

with

$$K = 1, \qquad U = M_{\mathrm{eff}}^2 + \mathrm{k}^2/a^2$$

where

$$\begin{aligned} \mathrm{I}: \quad M_{\mathrm{eff}}^2 &= M^2 + \frac{1}{3}\rho, \qquad m_{\mathrm{H}}^2 = M_{\mathrm{eff}}^2, \\ \mathrm{II}: \quad M_{\mathrm{eff}}^2 &= M^2 - p, \qquad m_{\mathrm{H}}^2 = M^2 + \rho \end{aligned}$$

m_{H} reduces to the Higuchi mass in the Einstein space limit.

Vector sector

4 auxiliary amplitudes are expressed in terms of two $\mathrm{V}_\pm$

$$W_\pm^+ = \frac{\mathrm{P}^2 m_\mathrm{H}^2 \,\dot{\mathrm{V}}_\pm}{m_\mathrm{H}^4 + \mathrm{P}^2(m_\mathrm{H}^2 - \epsilon/2)}, \quad W_\pm^- = \frac{\mathrm{P}^2 \,[m_\mathrm{H}^2 - \epsilon]\,\dot{\mathrm{V}}_\pm}{m_\mathrm{H}^4 + \mathrm{P}^2(m_\mathrm{H}^2 - \epsilon/2)},$$

(with $\epsilon = \rho + p$) and the effective action

$$I_{(1)} = \int (K\dot{\mathrm{V}}_\pm^2 - U\mathrm{V}_\pm^2)\, a^3 dt$$

with

$$\begin{aligned} K &= \frac{\mathrm{k}^2 m_\mathrm{H}^4}{m_\mathrm{H}^4 + (\mathrm{k}^2/a^2)(m_\mathrm{H}^2 - \epsilon/2)}, \\ U &= M_\mathrm{eff}^2\,\mathrm{k}^2 \end{aligned}$$

In Einstein spaces one has $m_\mathrm{H} = M_\mathrm{H}$ (Higuchi mass), vector modes do not propagate if $M_\mathrm{H} = 0$ (massless limit). Otherwise $m_\mathrm{H} \neq const. \Rightarrow$ they <u>always propagate</u>.

Scalar sector – auxiliary amplitudes

4 auxiliary amplitudes are expressed in terms of one single S

$$\begin{aligned}
S_+^- &= \frac{m_\mathrm{H}^2 - \epsilon}{m_\mathrm{H}^2}\, S_-^+ \,, \\
S_-^+ &= \frac{2}{m_\mathrm{H}^2}\, \left(\dot{S}_-^- + a^2 H S_+^+ \right) , \\
S_+^+ &= -\frac{1}{Ha^2}\, \dot{S}_-^- \\
&+ \frac{2Hm_\mathrm{H}^4 \mathrm{P}^2\, \dot{\mathrm{S}} + m_\mathrm{H}^6 \mathrm{P}^2\, \mathrm{S} - m_\mathrm{H}^4 (2\mathrm{P}^2 + 3m_\mathrm{H}^2) S_-^- / a^2}{2H^2 [3m_\mathrm{H}^4 + 2\mathrm{P}^2 (2m_\mathrm{H}^2 - \epsilon)]},
\end{aligned}$$

with $H = \dot{a}/a$ while

$$S_-^- = \mathcal{A}\, \dot{\mathrm{S}} + \mathcal{B}\, \mathrm{S}$$

where $\mathcal{A}, \mathcal{B}$ are complicated functions of the background scale factor a. The amplitude S fulfils one single master equation following from

Scalar sector – master equation

$$I_{(0)} = \int (K\dot{\mathrm{S}}^2 - U\mathrm{S}^2)\, a^3 dt$$

where the kinetic term

$$K = \frac{3\mathrm{k}^4 m_{\mathrm{H}}^4 (m_{\mathrm{H}}^2 - 2H^2)}{(m_{\mathrm{H}}^2 - 2H^2)[9m_{\mathrm{H}}^4 + 6(\mathrm{k}^2/a^2)(2m_{\mathrm{H}}^2 - \epsilon)] + 4(\mathrm{k}^4/a^4)(m_{\mathrm{H}}^2 - \epsilon)}$$

and the potential

$$\begin{array}{lllll} U/K & \rightarrow & M_{\mathrm{eff}}^2 & \text{as} & k \rightarrow 0 \\ U/K & \rightarrow & c^2\,(\mathrm{k}^2/a^2) & \text{as} & k \rightarrow \infty \end{array}$$

where c is the sound speed. There is only one DoF in the scalar sector (!!!) In the Einstein space one has $m_{\mathrm{H}} = M_{\mathrm{H}}$ and the scalar mode does not propagate if either $M_{\mathrm{H}} = 0$ (massless limit) or if $M_{\mathrm{H}}^2 = 2H^2$ (partially massless limit). In the generic case one has $m_{\mathrm{H}} \neq const$. and it always propagates.

No ghost conditions

$$\boxed{\lim_{\mathrm{k}\to\infty} K > 0}$$

with

$$\begin{aligned}
K_{(2)} &= 1, \\
K_{(1)} &= \frac{\mathrm{k}^2 m_{\mathrm{H}}^4}{m_{\mathrm{H}}^4 + (\mathrm{k}^2/a^2)(m_{\mathrm{H}}^2 - \epsilon/2)}, \\
K_{(0)} &= \frac{3\mathrm{k}^4 m_{\mathrm{H}}^4 (m_{\mathrm{H}}^2 - 2H^2)}{(m_{\mathrm{H}}^2 - 2H^2)[9m_{\mathrm{H}}^4 + 6(\mathrm{k}^2/a^2)(2m_{\mathrm{H}}^2 - \epsilon)] + 4(\mathrm{k}^4/a^4)(m_{\mathrm{H}}^2 - \epsilon)}
\end{aligned}$$

No tachyon conditions

$$\boxed{c^2 > 0}$$

with

$$\begin{aligned} c^2_{(2)} &= 1, \\ c^2_{(1)} &= \frac{M^2_{\rm eff}}{m^4_{\rm H}}\,(m^2_{\rm H} - \epsilon/2), \\ c^2_{(0)} &= \frac{(m^2_{\rm H} - \epsilon)[m^4_{\rm H} + (2H^2 - 4M^2_{\rm eff} - \epsilon)m^2_{\rm H} + 4H^2 M^2_{\rm eff}]}{3m^4_{\rm H}(2H^2 - m^2_{\rm H})}. \end{aligned}$$

Stability of the system

- Everything is stable if the background density is small, $\rho \leq M^2 M_{\rm Pl}^2$.
- Model II is stable for any ρ if $w = p/\rho < -2/5 \Rightarrow$ stable during inflation.
- Model I is stable during the inflation if the Hubble rate is not very high, $H < M$.
- Both models are always stable after inflation if $M \geq 10^{13}$ GeV.
- Both models are stable now if $M \geq 10^{-3}$ eV.
- Assuming that $X_{\mu\nu}$ couples only to gravity and hence massive gravitons do not have other decay channels, it follows that they could be a part of Dark Matter (DM) at present.

Backreaction

Self-coupled system

$$I = \frac{1}{2}\int \left(M_{\rm Pl}^2 R + X^{\nu\mu} E_{\mu\nu}\right)\sqrt{-g}\, d^4x.$$

Varying this with respect to the $X_{\mu\nu}$ and $g_{\mu\nu}$ yields

$$\begin{aligned} M_{\rm Pl}^2 G_{\mu\nu} &= T_{\mu\nu}, \\ E_{\mu\nu} &= 0, \end{aligned}$$

where the energy-momentum tensor $T_{\mu\nu}$ is rather complicated.
The only solution in the homogeneous and isotropic sector is found in model II and only for $M^2 < 0$: de Sitter with $\Lambda = -3M^2 > 0$
$\Rightarrow$ Massive gravitons in our model cannot mimic dark energy.

Other applications

- Holographic superconductors ... (?)
- Black hole solutions ?
- Boson star solutions ?

Spontaneous hair on black holes

Hairy black holes

- Einstein-Yang-Mills (+Higgs, dilaton) theory, Einstein-Skyrme (XX-th century)
- Scalars violating the energy conditions (phantom fields, negative potentials) (XXI-st century)
- Non-minimally coupled scalars (Horndeski) (XXI-st century)
- Non-linear massive gravity (XXI-st century)
- Spinning scalar clouds (XXI-st century)

/M.S.V. arXiv:1601.08230/

Superradiance

- Incident waves with $\omega < m\Omega_{\mathrm{H}}$ are amplified by a spinning black hole /Zel'dovich 1971/, /Starobinsky 1972/, /Bardeen, Press, Teukolsky 1972/
- If the black hole is surrounded by mirror walls, the field will be trapped inside the walls but its amplitude will grow – "black hole bomb" /Press, Teukolsky 1972/
- If the field has a mass μ then its modes with $|\omega| < \mu$ cannot escape to infinity and will stay close to the black hole /Damour, Deruelle, Ruffini 1976/. Such modes will be amplified but also absorbed by the black hole.

Black hole hair via superradiance

- If the amplification and absorption rates of massive modes are equal, this will lead to non-trivial stationary field clouds around spinning black holes.
- This suggests that spinning black holes may support massive hair and moreover, spontaneously grow it.
- First confirmation of this scenario – scalar Kerr clouds = spinning black holes with massive complex scalar field /Herdeiro, Radu, 2014/.
- Next – spinning black holes with massive complex vector field /Herdeiro, Radu, Runarsson 2016/.

Black hole hair via superradiance

- First confirmation of the spontaneous growth phenomenon – growth of massive complex vector field /East, Pretorius 2017/. As the *supperadiance rate increases with spin*, the vector massive hair grows faster than the scalar one – easier to simulate.

- However, the tensor hair should grow still faster. This suggest there should be spinning black holes with complex massive graviton hair. Complexification – replacing

$$X^{\nu\mu} E_{\mu\nu} \rightarrow \bar{X}^{\nu\mu} E_{\mu\nu} + X^{\nu\mu} \bar{E}_{\mu\nu}$$

in the action

$$I = \frac{1}{2} \int \left(M_{\mathrm{Pl}}^2 R + X^{\nu\mu} E_{\mu\nu} \right) \sqrt{-g}\, d^4x.$$

Summary of results

- The consistent theory of massive gravitons in arbitrary spacetimes presented in the form simple enough for practical applications.
- The theory is described by a non-symmetric rank-2 tensor whose equations of motion imply six algebraic and five differential constraints reducing the number of independent components to five.
- The theory reproduces the standard description of massive gravitons in Einstein spaces.
- In generic spacetimes it does not show the massless limit and always propagates five degrees of freedom, even for the vanishing mass parameter.
- The explicit solution for a homogeneous and isotropic cosmological background shows that the gravitons are stable, hence they may be a part of Dark Matter.
- An interesting open issue – possible existence of stationary black holes with massive graviton hair.