Orsay, 22 **Nov** 2010

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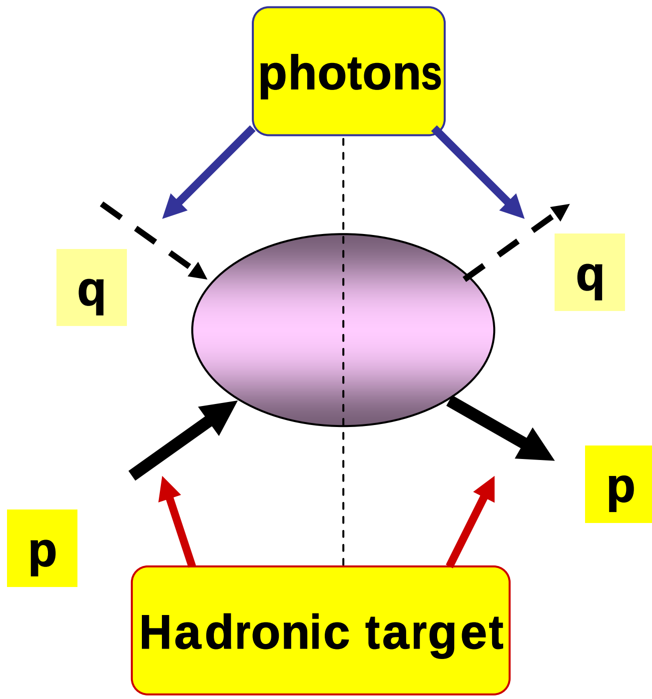
Requirements for initial parton densities following from factorization

talk based on results obtained in collaboration with M. Greco and S.I. Troyan

$$W_{\mu\nu} =$$



$$W_{\mu\nu} = \sum P^{a}_{\mu\nu} \; f_a(x, Q^2)$$

x = Q^2=2pq

Projection operators

Structure funcrions

Φορ εξαμπλε:

$$W^{unpol}_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x, Q^2) + \frac{1}{pq} \left(p_\mu - q_\mu \frac{pq}{q^2} \right) \left(p_\nu - q_\nu \frac{pq}{q^2} \right) F_2(x, Q^2)$$

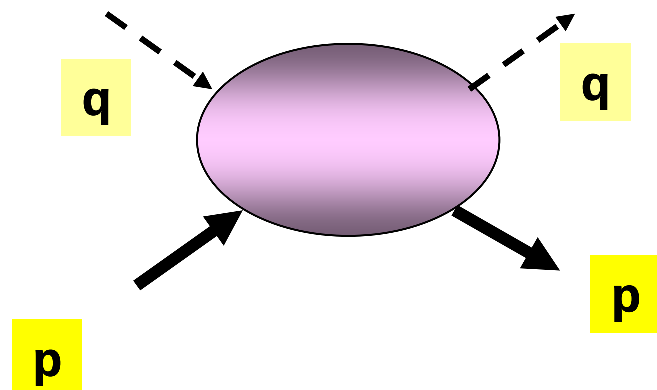
Optical Theorem:

forward Compton scattering amplitude

$$W_{\mu\nu} = \frac{1}{\pi} \mathrm{Im}\, A_{\mu\nu} \quad \Rightarrow \quad A_{\mu\nu} = \sum P^r_{\mu\nu} A_r(x, Q^2)$$

projection operators | invariant amplitudes

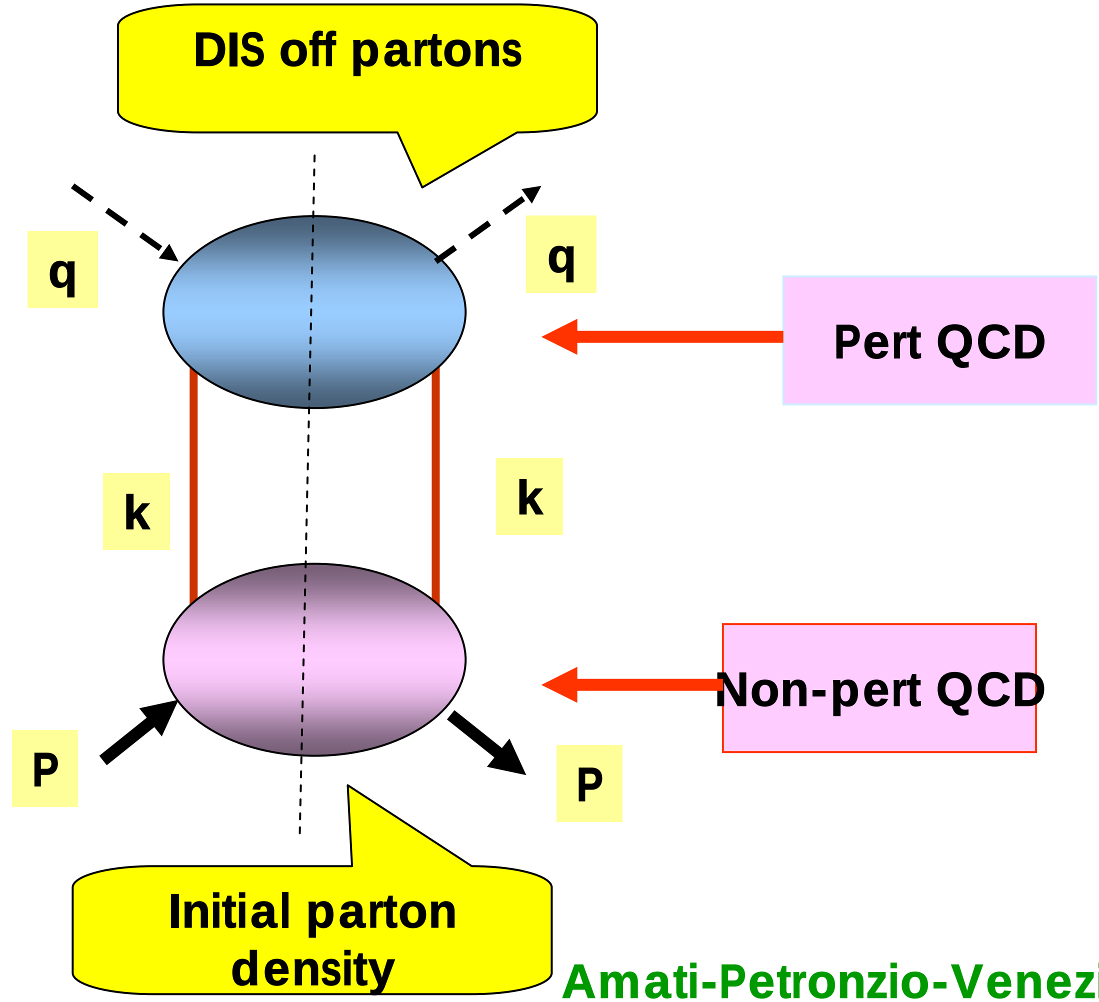
$$W_{\mu\nu} = \frac{1}{\pi} \mathrm{Im}$$



$$f_r = \frac{1}{\pi} \mathrm{Im}\, A_r$$

Factorization:

$$W_{\mu\nu} = \sum$$

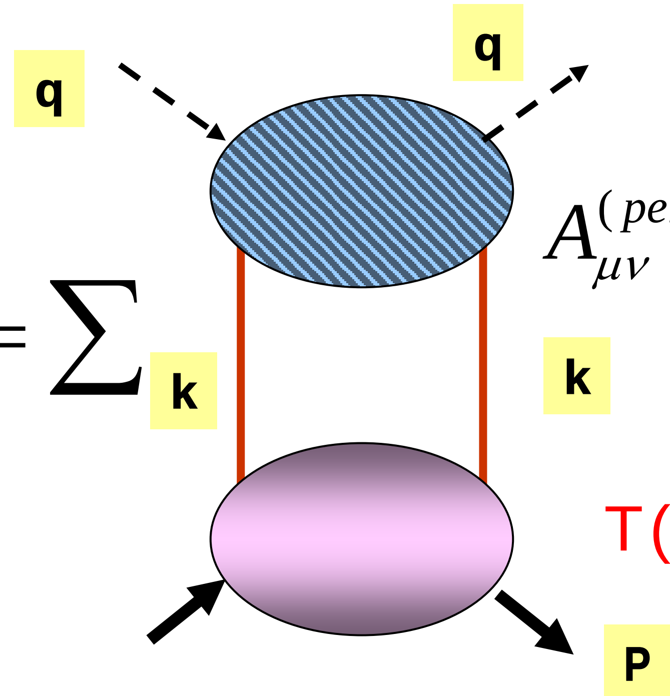


Collinear factorization (for DGLAP) Amati-Petronzio-Veneziano, Efremov-Radyushkin, Libby-Sterman, Brodsky-Lepage,..

k_T –factorization (when BFKL is used) Catani-Ciafaloni-Hautmann

Amplitude of forward Compton scattering

$$A_{\mu\nu}(q,p) = \sum_{k}$$



$$A^{(pert)}_{\mu\nu}(q,k)$$

Pert QCD

T(k,p, m,S)

Non-pert QCD

Target mass and spin

$$A_{\mu\nu}(q,p) = \int \frac{d^4k}{(2\pi)^4} A^{(pert)}_{\mu\nu}(q,k) \frac{1}{(k^2)^2} T(k,p)$$

K acts as IR cut-off for IR-sensitive contributions

$$A(q,p) = \int \frac{d^4k}{(2\pi)^4} A^{(pert)}(q,k) \frac{1}{(k^2)^2} T(k,p)$$

any of the invariant amplitudes

Pert contribution

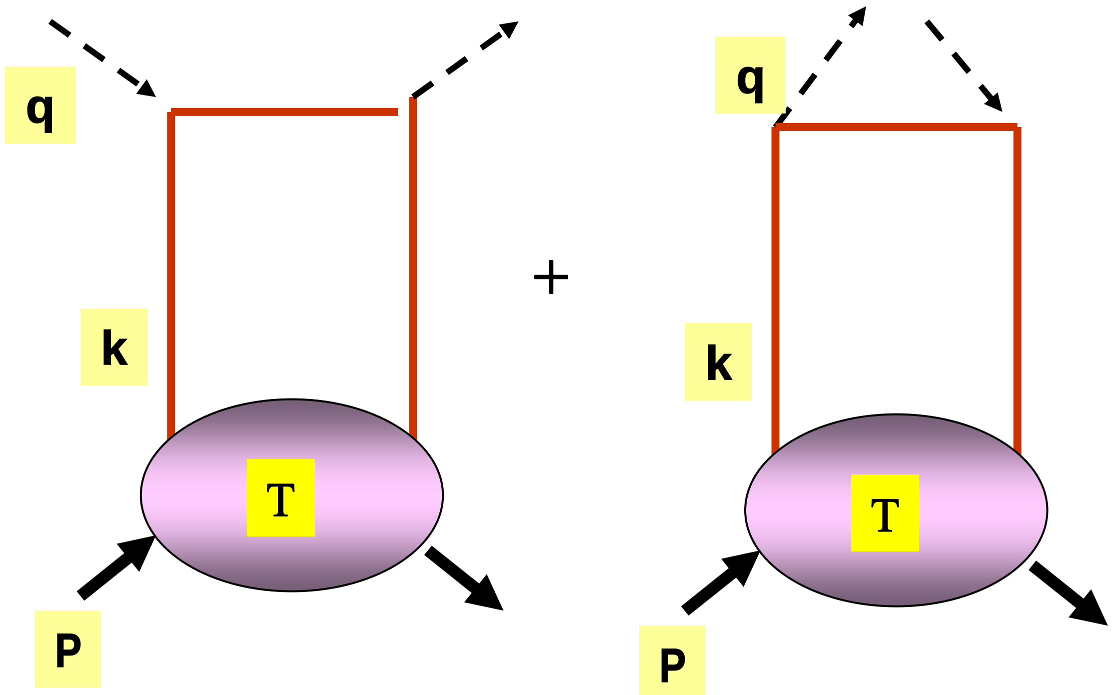
Non-Pert term

Calculated either in fixed orders or with Evolution Equations

Unknown from theoretical grounds

integration over k should be free of UV and IR divergences → **requirements for T**

Born approximation



UV behavior: **At Euclidean k** $d^4k = d\Omega k^3 dk$

So, at large k $A \sim \int dk \frac{k^3}{k^3} T(k)$ → $T \sim k^{-1-h}$ **h** > 0

In Pert QCD T is gluon propagator: $T = 1/k^2$

In Minkowsky space:

Sudakov parameterization

$$k = -\alpha\ p + \beta\,(q + xp) + k_{\perp}$$

So that

$$k^2 = -\alpha\beta\ w - k_{\perp}^2,\ \ 2\,pk = -\alpha w,\ \ 2qk = (\beta + x\alpha)w$$

w= 2pq

$$A_{Born}^{(pert)} = \frac{\gamma_\nu(\hat{q} + \hat{k})\gamma_\mu}{(q+k)^2} + \frac{\gamma_\mu(-\hat{q} + \hat{k})\gamma_\nu}{(q-k)^2}$$

At large α

$$\int d\alpha \, \frac{1}{\overset{\frown}{k}} A^{(pert)}_{Born}(q,k) \frac{1}{\overset{\frown}{k}} T(k,p) \sim \int d\alpha \, \frac{\alpha^3}{\alpha^3} T(k,p)$$

$$T \sim \alpha^{-1-h}$$

$h > 0$

$$T(p,k) = T((p+k)^2, k^2) = T(w\alpha, (w\alpha\beta + k_\perp^2))$$

Beyond the Born approximation

$$A(q,p) = \int \frac{d^4k}{(2\pi)^4} A^{(pert)}(q,k) \frac{B(k)}{k^2 k^2} T(k,p)$$

perturbative

non-pert

where

$$B(k) \approx (\alpha^2 + \beta^2) w + k_\perp^2$$

In the first place consider Perturbative amplitude $A^{(pert)}$

$$A^{(pert)}(q,k) = A^{(pert)}((q+k)^2, \underline{k^2}, q^2)$$

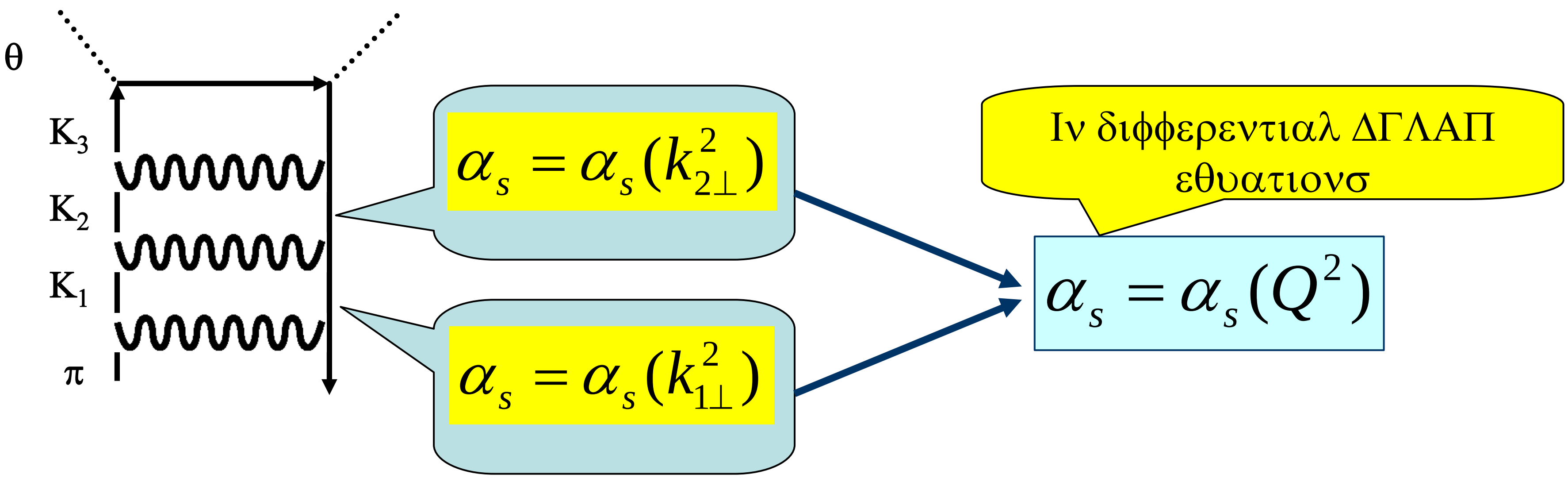
k^2 plays the role of IR cut-off for IR-dependent terms in $A^{(pert)}$

Sources for k^2 –dependence:

A: k^2 acts as IR cut-off to regulate IR singularities in integrals, i.e. acts as the lowest integration limit

B. Treatment of QCD coupling

Parameterization of QCD coupling



However, this parametrization is approximation. Analysis shows that in DGLAP expressions $\alpha_s(k_\perp^2)$ should be replaced by a more complicated expression: $\alpha_s(k_\perp^2) \rightarrow \alpha_s^{eff}$

$$\alpha_s^{eff} = \alpha_s(\mu^2) + \frac{1}{\pi b}\left[\arctan\left(\frac{\pi}{\ln(k_\perp^2/\beta\Lambda^2)}\right) - \arctan\left(\frac{\pi}{\ln(\mu^2/\Lambda^2)}\right)\right]$$

IR –dependent terms

Ermolaev - Troyan

μ **IR cut-off. In order to be in Perturbative regime**
It should be large enough: $\mu >> \Lambda$

when $\ln(\mu^2/\Lambda^2) >> \pi$ → $\arctan x \approx x$

$$\alpha_s^{eff} \approx \alpha_s(\mu^2) + \frac{1}{\pi b}\left[\frac{\pi}{\ln(k_\perp^2/\beta\Lambda^2)} - \frac{\pi}{\ln(\mu^2/\Lambda^2)}\right] = \alpha_s\left(k_\perp^2/\beta\right)$$

practical estimate for condition

$$\mu^2 >> \Lambda^2 e^{\pi} \approx 23\,\Lambda^2$$

$$R(\mu) = \frac{(1/\pi)\arctan(\pi/l) - 1/l}{\arctan(1/l)}, \qquad l = \ln(\mu^2/\Lambda^2)$$

$R = 5\%$	→	$\mu^2 = \Lambda^2\, 2.810^3 \approx 28\,\mathrm{GeV}^2$
$R = 10\%$	→	$\mu^2 = \Lambda^2\, 243 \approx 2.4\,\mathrm{GeV}^2$
$R = 50\%$	→	$\mu^2 = \Lambda^2\, 8.72 \approx 0.87\,\mathrm{GeV}^2$

minimal option

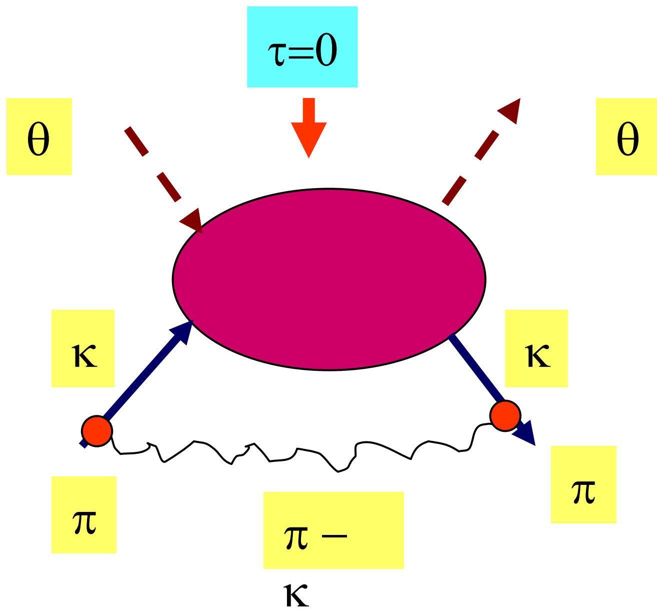
at $\Lambda = 0.1\,\mathrm{GeV}$

close to conventional option 1 GeV2 !!

More realistic would be to change 0.1 GeV for 0.5 which causes increase of μ

PROOF

Parameterization of α_s in Regge kinematics



$$M_s = \frac{i}{4\pi^2}\int d\alpha d\beta dk_\perp^2 M(2qk, Q^2, k^2)\frac{sk_\perp^2}{(k^2+i\varepsilon)^2}\frac{\alpha_s((p-k)^2)}{(p-k)^2+i\varepsilon}$$

$$k = -\alpha\,(q+xp)+\beta\;p+k_\perp$$

$$k^2 = -s\alpha\beta - k_\perp^2, \qquad (p-k)^2 = s\alpha - s\alpha\beta - k_\perp^2$$

$$m^2 \equiv (p-k)^2 \quad \Rightarrow \quad s\alpha = \frac{m^2 + k_\perp^2}{1-\beta}$$

$$M_s = \frac{i}{4\pi^2} \int d\beta dk_\perp^2 (1-\beta) I(s, Q^2, \beta, k_\perp^2)$$

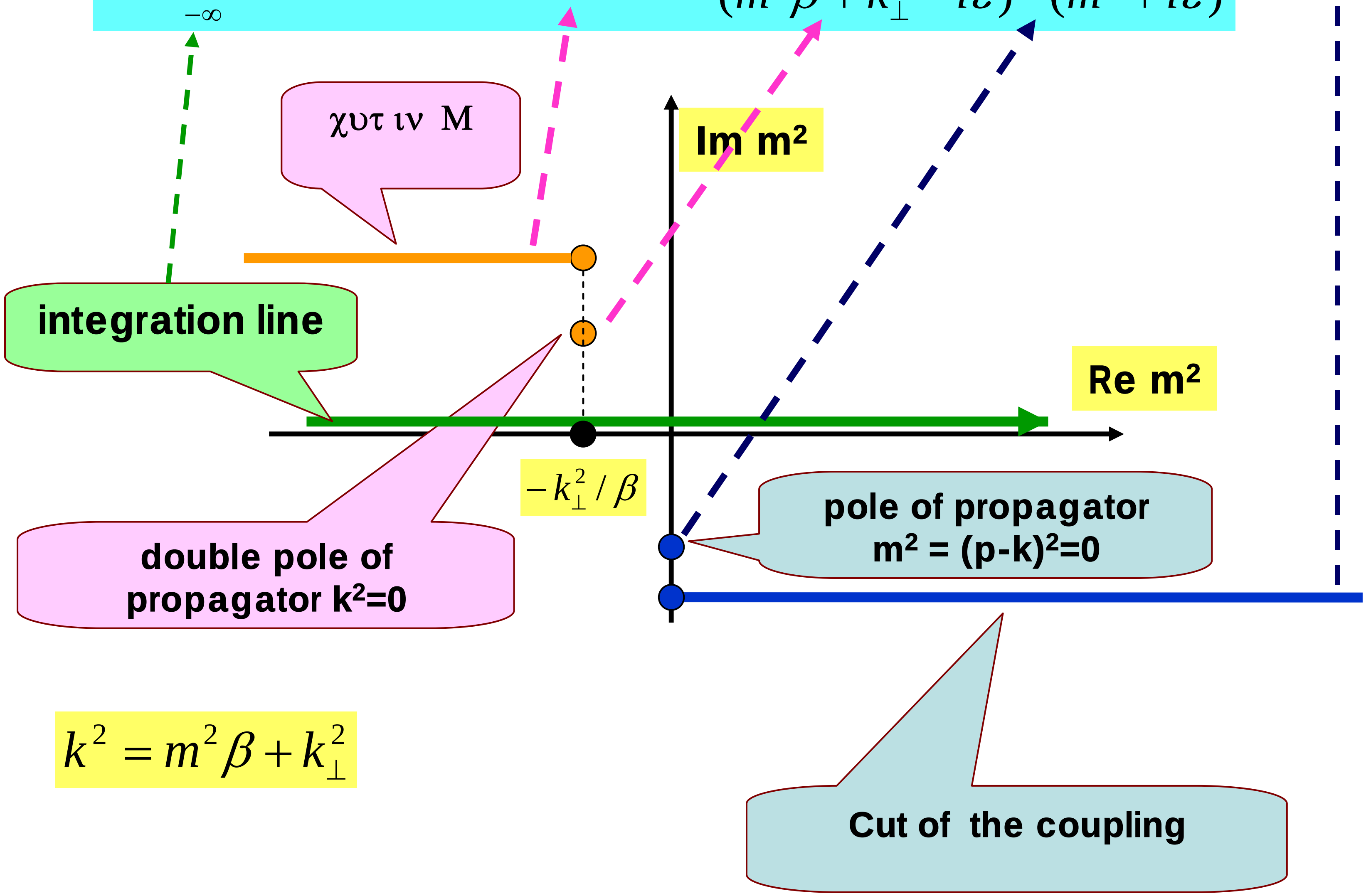
ωηερε

$$I = \int_{-\infty}^{\infty} dm^2 M(s, Q^2, (m^2\beta + k_\perp^2)) \frac{k_\perp^2}{(m^2\beta + k_\perp^2 - i\varepsilon)^2} \frac{\alpha_s(m^2)}{(m^2 + i\varepsilon)}$$

Before integrating over m², let us study singularities of the integrand

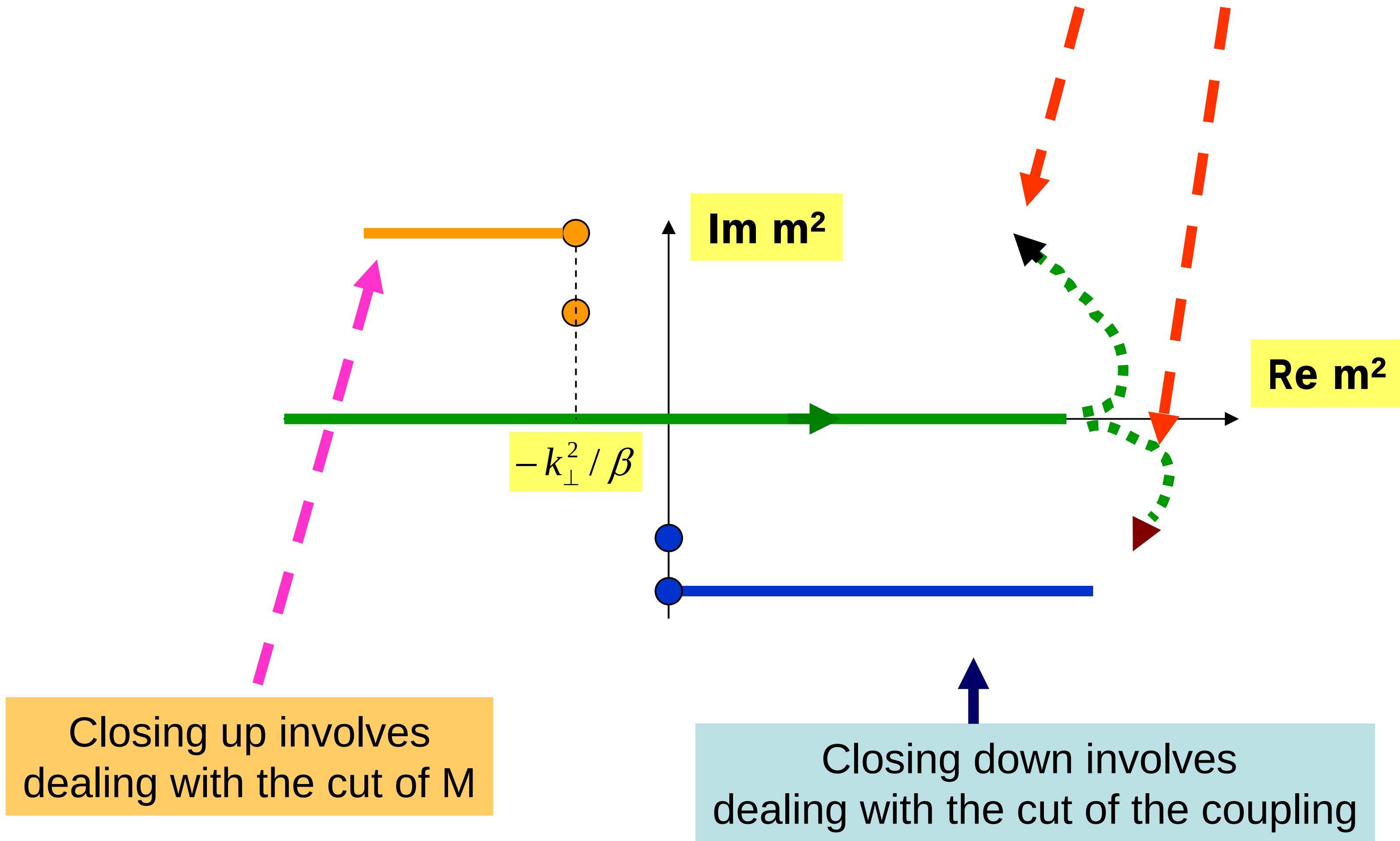
singularities of the integrand in μ^2

$$I = \int_{-\infty}^{\infty} dm^2 M(s, Q^2, (m^2\beta + k_\perp^2)) \frac{k_\perp^2}{(m^2\beta + k_\perp^2 - i\varepsilon)^2} \frac{\alpha_s(m^2)}{(m^2 + i\varepsilon)}$$



$$k^2 = m^2\beta + k_\perp^2$$

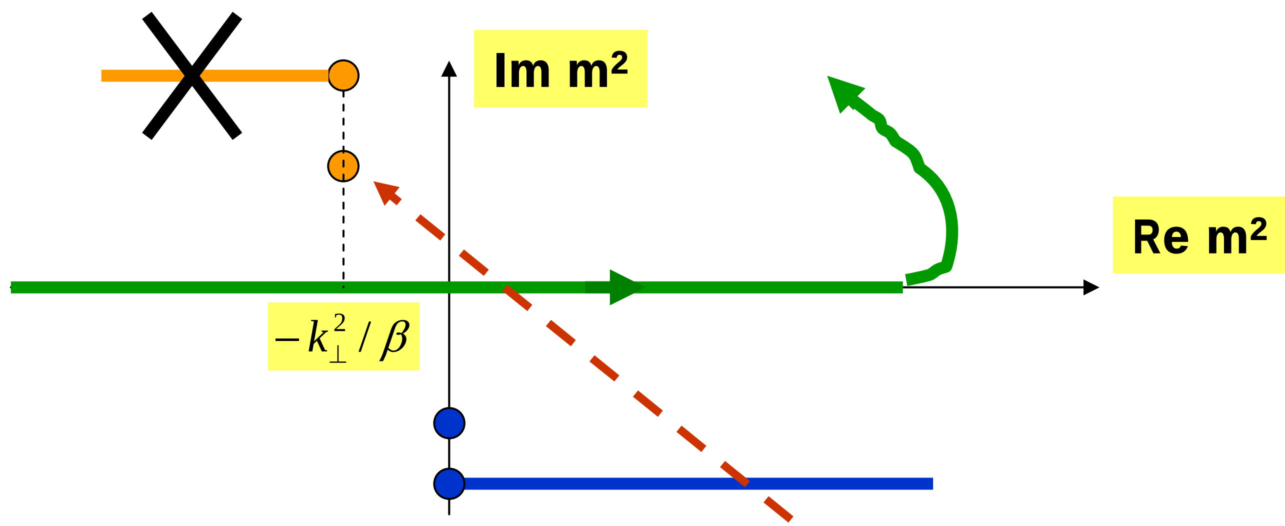
It is convenient to perform integration over m^2 with using Cauchy theorem.
the integration contour has to be closed either up or down



Approximation: $M(s,..,k^2) = M(s,..,m^2\beta + k_\perp^2) \approx M(s,..,k_\perp^2)$

Assumption: $m^2\beta << k_\perp^2$

no cut in μ^2



allows to close up the contour and deal with the pole only:

$m^2 = -k_\perp^2 / \beta$

$\alpha_s = \alpha_s(-k_\perp^2 / \beta)$

CONTRADICTION between the assumption and calculation:

$m^2 \beta << k_\perp^2$

assumption

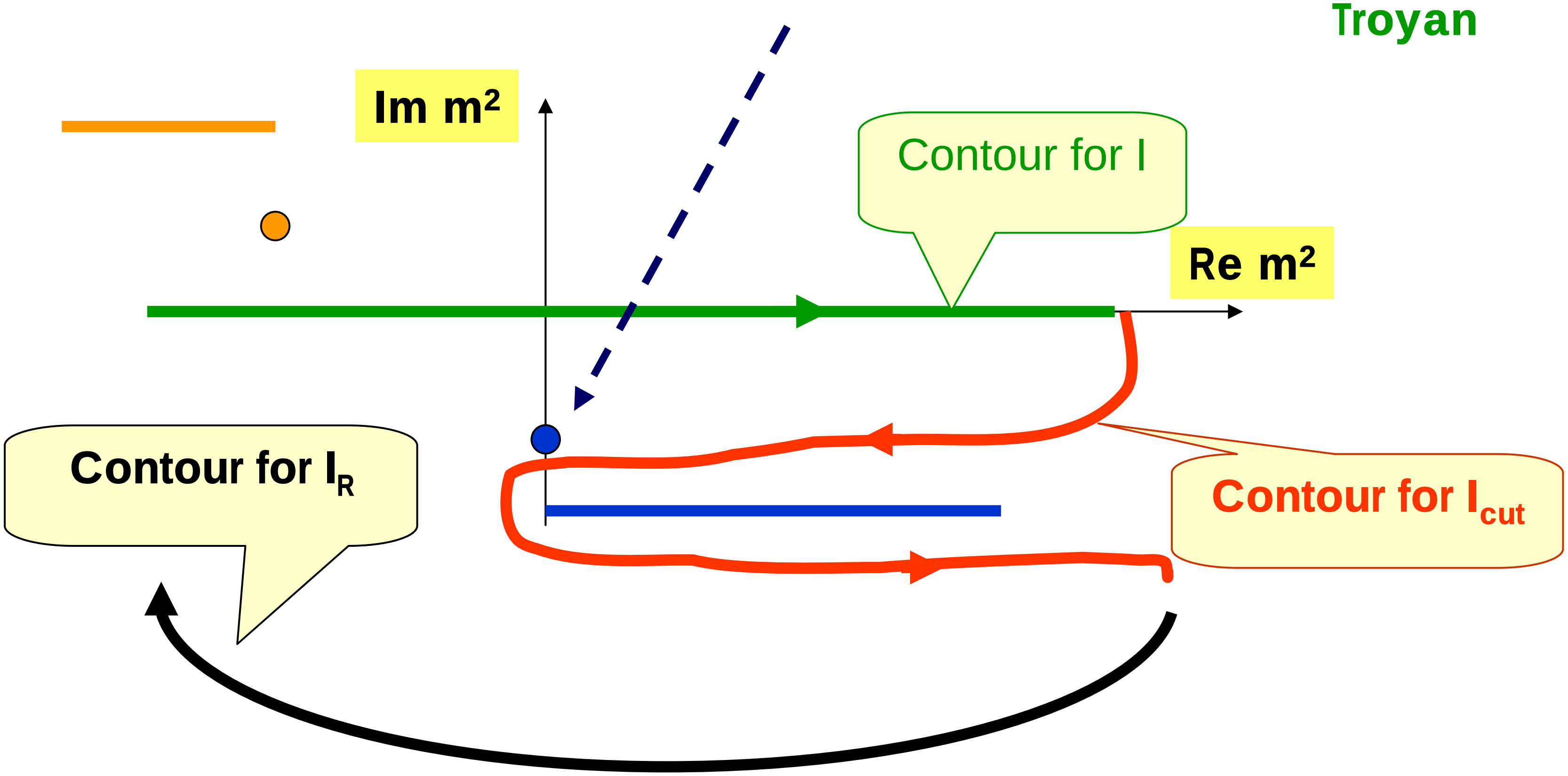
$m^2 \beta + k_\perp^2 = 0$

residue in the pole

Therefore the result $\alpha_s = \alpha_s(-k_\perp^2 / \beta)$ **should be revised**

we close the contour down to avoid dealing with singularities of M

Ermolaev-Troyan



Cauchy theorem:

$$I_C = I + I_R + I_{cut} = -2\pi i \; res \quad \text{in the pole at } m^2 = 0$$

$$I_C = I + I_R + I_{cut} = -2\pi i \frac{(1-\beta)}{k_\perp^2} M(s, Q^2, -k_\perp^2/(1-\beta))\alpha_s(\mu^2)$$

$$\mu^2 >> \Lambda^2$$

Introduced to regulate IR region

$I_R \to 0$ when $R \to \infty$

$$I_{cut} = -2i \int_{\mu^2}^{\infty} dm^2 M(s, Q^2, (m^2\beta + k_\perp^2)) \frac{(1-\beta)k_\perp^2}{(m^2\beta + k_\perp^2 + i\varepsilon)^2} \frac{\mathrm{Im}\,\alpha_s(m^2)}{(m^2 + i\varepsilon)}$$

Assumption: **if we assume that the essential region is** $k_\perp^2 >> m^2\beta$

$$I_{cut} \approx -2i \frac{\pi}{b} \frac{(1-\beta)}{k_\perp^2} M(s, Q^2, k_\perp^2) \int_{\mu^2}^{k_\perp^2/\beta} \frac{dm^2}{m^2} \frac{1}{[\ln(m^2/\Lambda^2) + \pi^2]}$$

$$I = -2\pi i \frac{(1-\beta)}{k_\perp^2} M(s, Q^2, k_\perp^2)\alpha_s^{eff},$$

$$\alpha_s^{eff} = \alpha_s(\mu^2) + \frac{1}{\pi b}\left[\arctan\left(\frac{\pi}{\ln(k_\perp^2/\beta\Lambda^2)}\right) - \arctan\left(\frac{\pi}{\ln(\mu^2/\Lambda^2)}\right)\right]$$

$$= \alpha_s(\mu^2) + \frac{1}{\pi b}\left[\arctan\left(\pi b\, \alpha_s(k_\perp^2/\beta)\right) - \arctan\left(\pi b\, \alpha_s(\mu^2)\right)\right]$$

$\ln(\mu^2/\Lambda^2) >> \pi$ → $\alpha_s^{eff} \approx \alpha_s(k_\perp^2/\beta)$

When additionally $x \sim 1$ → $\beta \sim 1$ → $\alpha_s(k_\perp^2/\beta) \approx \alpha_s(k_\perp^2)$

DGLAP region (for $x \sim 1$)

DGLAP parameterization (for $\alpha_s(k_\perp^2/\beta) \approx \alpha_s(k_\perp^2)$)

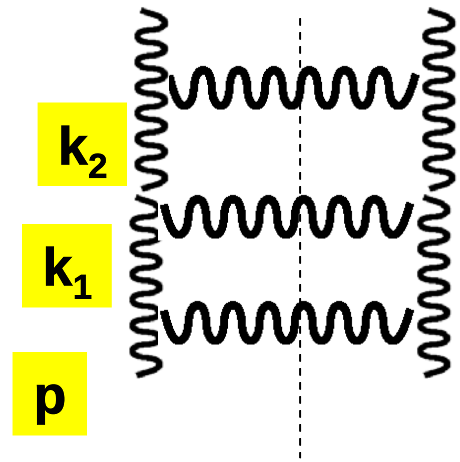
Therefore $\alpha_s^{eff} \approx \alpha_s(k_\perp^2)$ only when $x \sim 1$ and $\mu^2 >> \Lambda^2 e^{\pi}$

Treatment of gluon ladders: vertical propagators are IR-divergent

First loop

DGLAP	LL
$\int_{\mu^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}$	$\int_{\mu^2}^{w} \frac{dk_\perp^2}{k_\perp^2}$

identical IR -dependence



Second loop

DGLAP	LL
$\int_{k_{1\perp}^2}^{Q^2} \frac{dk_{2\perp}^2}{k_{2\perp}^2} \int_{x}^{1} \frac{d\beta_2}{\beta_2}$	$\int_{k_{1\perp}^2}^{w} \frac{dk_{2\perp}^2}{k_{2\perp}^2} \int_{k_{1\perp}^2/w}^{\beta_1} \frac{d\beta_2}{\beta_2} + \int_{\mu^2}^{k_{1\perp}^2} \frac{dk_{2\perp}^2}{k_{2\perp}^2} \int_{k_{1\perp}^2/w}^{\beta_1 k_{2\perp}^2/k_{1\perp}^2} \frac{d\beta_2}{\beta_2}$

no IR -dependence

DGLAP-like term

new term: IR -dependence

the same in higher loops

There are two different situations beyond Born approximation:

$$\mathrm{Im}\, A \sim F_1^{\sin glet}$$ ← vacuum numbers in t-channel

Refer as singlet A_S

$$\mathrm{Im}\, A \sim F_1^{NS}, g_1^{S}, g_1^{NS}$$ ← non-vacuum numbers in t-channel

Refer as non-singlet A_{NS}

singlet and non-singlet have different IR-sensitive contributions:

$$A_S^{(pert)} = \left(\frac{w\beta}{k^2}\right) M_S \left(\ln(w\beta / k^2), \ln(Q^2 / k^2)\right)$$

$$A_{NS}^{(pert)} = M_{NS} \left(\ln(w\beta / k^2), \ln(Q^2 / k^2)\right)$$

Perturbative contributions M are different for different amplitudes and in different approaches but their arguments are always the same

$$M = \sum C_{kl} \ln^k (w\beta / k^2) \ln^l (Q^2 / k^2)$$ **+ non-logarithmic contributions**

Do not involve powers α^n

$$k^2 = -\alpha\beta\, w - k_\perp^2$$

$$A_S = \int dk_\perp^2 \frac{d\beta}{\beta} d\alpha \left(\frac{w\beta}{\underline{k^2}} \right) M_S \left(\ln(w\beta / k^2), \ln(Q^2 / k^2) \right) \frac{B(k)}{(k^2)^2} T_S (w\alpha, k^2)$$

$$A_{NS} = \int dk_\perp^2 \frac{d\beta}{\beta} d\alpha M_{NS} \left(\ln(w\beta / k^2), \ln(Q^2 / k^2) \right) \frac{B(k)}{(k^2)^2} T_{NS} (w\alpha, k^2)$$

Now let us integrate over α neglecting α -dependence in logs

$A_S \Leftrightarrow \int d\alpha \frac{\alpha^2}{\underline{\alpha}\, \alpha^2} T_S(\alpha)$

$A_{NS} \Leftrightarrow \int d\alpha \frac{\alpha^2}{\alpha^2} T_{NS}(\alpha)$

to arrive at

Obligatory for integrability

$$T_S \sim \alpha^{-h}, \qquad T_{NS} \sim \alpha^{-1-h}$$

Application to DIS structure functions

$$f_S = \int dk_\perp^2 \frac{d\beta}{\beta} w\beta \; f_S^{(pert)}\left(\ln(w\beta / k^2), \ln(Q^2 / k^2)\right) \int d\alpha \frac{B(k)}{(k^2)^3} \mathrm{Im}\, T_S(w\alpha, k^2)$$

stands for singlet F_1 only

$$k^2 = -\alpha\beta\, w - k_\perp^2$$

$$f_{NS} = \int dk_\perp^2 \frac{d\beta}{\beta} \; f_{NS}^{(pert)}\left(\ln(w\beta / k^2), \ln(Q^2 / k^2)\right) \int d\alpha \frac{B(k)}{(k^2)^2} \mathrm{Im}\, T_{NS}(w\alpha, k^2)$$

any of F_1^{NS}; F_2; g_1^{S}; g_1^{NS}; g_2

No factorization in $\alpha \quad \beta$

Factorization is only when $\quad \alpha\beta\, w << k_\perp^2$

When it is accepted, we arrive at the standard expressions

Singlet

$$f_S = \int \frac{dk_\perp^2}{k_\perp^2} \frac{d\beta}{\beta} \left(\frac{w\beta}{k_\perp^2} \right) f_S^{(pert)} \left(\ln(w\beta / k_\perp^2), \ln(Q^2 / k_\perp^2) \right) \Phi_S(\beta, k_\perp^2)$$

Non-singlet

$$f_{NS} = \int \frac{dk_\perp^2}{k_\perp^2} \frac{d\beta}{\beta} f_{NS}^{(pert)} \left(\ln(w\beta / k_\perp^2), \ln(Q^2 / k_\perp^2) \right) \Phi_{NS}(\beta, k_\perp^2)$$

Where the singlet and non-singlet initial parton densities are

$$\Phi_S = \int_{k_\perp^2/w}^{k_\perp^2/w\beta} d\alpha \, \mathrm{Im} T_S(w\alpha, k_\perp^2)$$

$$\Phi_{NS} = \int_{k_\perp^2/w}^{k_\perp^2/w\beta} d\alpha \, \mathrm{Im} T_{NS}(w\alpha, k_\perp^2)$$

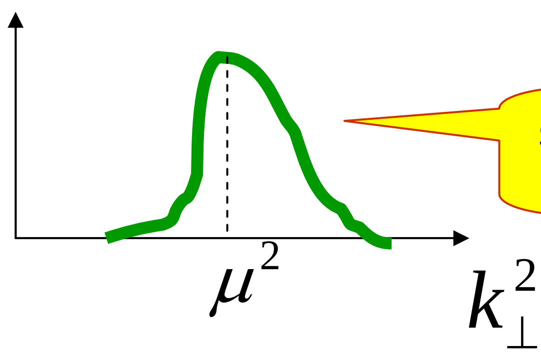
$$T_S \sim \alpha^{-h} \quad \rightarrow \quad \Phi_S \sim \beta^{-1+h}$$

$$T_{NS} \sim \alpha^{-1-h} \quad \rightarrow \quad \Phi_{NS} \sim \beta^{h}$$

h>0

Transition to DGLAP: Collinear factorization

$$\Phi = \Phi(\beta, k_\perp^2)$$



sharp maximum

For instance: $\Phi = \tilde{\Phi}(\beta, k_\perp^2)\,\delta(k_\perp^2 - \mu^2)$

$$f = \int \frac{d\beta}{\beta} f^{(pert)}\left(\ln(w\beta/\mu^2), \ln(Q^2/\mu^2)\right)\Phi\ (\beta, \mu^2)$$

Explicit dependence on μ^2 even after convoluting pert and non-pert

However, there is no μ^2 dependence in the case of DGLAP because DGLAP collects leading logs of Q^2 only, Sub-leading logs will be μ^2 -dependent

$$\ln^n(Q^2/\mu_1^2) = \ln^n(Q^2/\mu_2^2)$$

+ sub-leading

Comparison to standard DGLAP fits:

Altarell-Bal-Forte-Ridolfi, Leader-Sidorov-Stamenov, Blumlein-Botcher, Hirai,…

Typical expressions:

parameters **a,b,c,d** > 0

$$\delta q,\ \delta g = x^{-a}(1-x)^{b}(1+cx^{d})$$

singular factor

regular terms

$$\Phi_S \sim \beta^{-1+h}$$

→ singular factor **a** **<1** for F_1 singlet

integrability

h>0

$$\Phi_{NS} \sim \beta^{h}$$

→ excludes singular factors for all other structure functions

However in practice these requirements are violated

Reason why singular factors are necessary in DGLAP at small ξ

Most important at small **x**

DGLAP does not include resummations of $\sim \ln^k(1/x)$, so without singular factors x^{-a} DGLAP expressions grow too slowly to match experiment

Factors x^{-a} bring the appropriate growth at small **x**

They mimic resummation of $\sim \ln^k(1/x)$ and eventually, at **x -> 0** they change the classic DGLAP asymptotics

$$f \sim \exp\left[\sqrt{\ln(1/x)}\right]$$

for the Regge one

$$f \sim x^{-a}$$

When the resummation is accounted for, they should be dropped, which simplifies fits

Conclusion A (perturbative QCD)

A1. Strictly speaking, the QCD coupling in the Bethe-Salpeter equations for the scattering amplitudes/parton densities/structure functions cannot be factorized

A2. Factorization of the coupling is approximation and leads to converting α_s into α_s^{eff}

A3. For the hard processes we confirm known result:

$$\alpha_s^{eff} = \alpha_s(k_\perp^2/(1-\beta)) \approx \alpha_s(k_\perp^2)$$

A4. For the Regge processes and evolution equations

$$\alpha_s^{eff} = \alpha_s(\mu^2) + \frac{1}{\pi b}\left[\arctan\left(\pi b \alpha_s(k_\perp^2/\beta)\right) - \arctan\left(\pi b \alpha_s(\mu^2)\right)\right]$$

and it explicitly depends on the IR cut-off/starting point of the Q^2 –evolution i.e. exhibits really non-trivial IR -dependence

A5. Alternatively, one can use Mellin transform

CONCLUSION B (Non-Perturbative part)

Integrability of forward Compton amplitudes imposes the following restrictions on DGLAP fits for initial parton densities:

B1. Singular factors x^{-a} can be used in fits for singlet F_1 only, providing $a<1$

B2. Singular factors should not be used for all other structure functions. Instead, one should use total resummation of

B3. Necessity to use singular factors is a good indication that important logs of x are missing from theoretical expressions

In general, the use of collinear factorization brings dependence factorization scale. However, DGLAP -expressions for structure functions do not depend on it because DGLAP deals with leading logs of Q^2 and neglect sub-leading logs